Chapter 1

Kinematics

1.1 Introduction

Kinematics is the branch of mechanics that describes the motion of a body, which is the apparent change in its position with time, while ignoring the agents causing this motion. Most of the bodies studied by physicists are in motion. Motion occurs at all scales of the universe, from particles like electrons, protons, and neutrons that make up atoms to galaxies. It is crucial to define motion properly to understand many phenomena we observe around us.

A body can have different types of motion:

- Translational motion: e.g., the motion of a car on a road.
- Rotational motion: e.g., the rotation of the Earth on its axis.
- Vibrational motion: e.g., small oscillations in a mass-spring system.
- A combination of several of these types of motion.

1.1.1 Reference Frame

Rest and motion are relative concepts. For instance, an observer A who is at rest sees a tree in a fixed position, while observer B, who is the driver of a moving car nearby, sees the tree moving backward.

This example illustrates that describing motion requires specifying the nature of the observer. In physics, the study of motion is carried out by replacing the observer with a coordinate system, also known as a reference frame. A reference frame can be either fixed or moving: the one associated with A is fixed, and the one associated with B is moving.

To express the concepts of rest and motion relative to a reference frame, consider an orthonormal coordinate system $R(O, x, y, z)$ in which the position of a body, M(x, y, z), is determined. The body is at rest with respect to this reference frame if its coordinates remain constant over time. However, if at least one of them changes, the body is in motion relative to R.

1.2 I.2. Concept of a Material Point

The motions of bodies are often very complex. When studying the motion of an object, if we only consider its position, we can simplify the problem by representing the object as a material point with the same mass, located at its center of gravity. This simplification neglects any rotational effects of the solid object on itself or its spatial extent.

Example: Mass-string-pulley system in Figure II.1:

- Point A can be reduced to a material point;

Point B and the string cannot be reduced to material points.

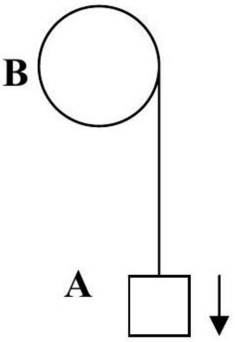


Figure 1.1

1.2.1 Trajectory

The trajectory is the geometric locus of successive positions occupied by the material point over time, relative to the chosen reference system.

The trajectory can be a physical reality (such as a road, railway track, etc.) or a physical concept not necessarily materialized (e.g., the trajectory of a projectile).

1.3 RECTILINEAR MOTION

The choice of rectilinear motion is motivated by its simplicity: it is easy to describe and can be represented by simple equations.

1.3.1 Definition

In this type of motion, the trajectory followed is a straight line. The reference frame can be reduced to an origin O and an axis Ox aligned with the trajectory.

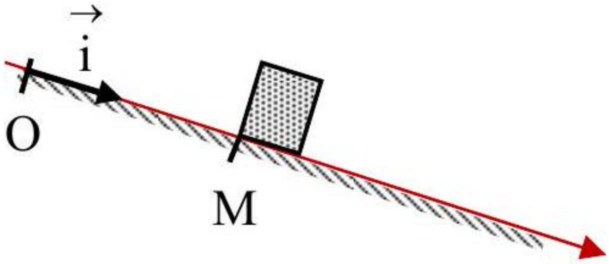


Figure 1.2

The position M of the object is represented by the position vector:

$$\overrightarrow{\mathrm{OM}} = x\,\vec{i}$$

1.3.2 Space Diagram

The position of point M depends on time. Therefore, at any instant ”t”, it can be represented by the vector:

$$\overrightarrow{\mathrm{OM}}(t) = x(t)\,\overrightarrow{i}$$

The relation $x = f(t)$ is the equation for the motion over time. For example, $x = \frac{1}{2}gt^2$ describes the free fall of an object dropped from the origin O of a vertically oriented axis.

The graph of $x(t)$ is called the space diagram.

Note: The space diagram is not necessarily a straight line, even in the case of rectilinear motion. It should not be confused with the trajectory.

Example: Space diagram for free fall:

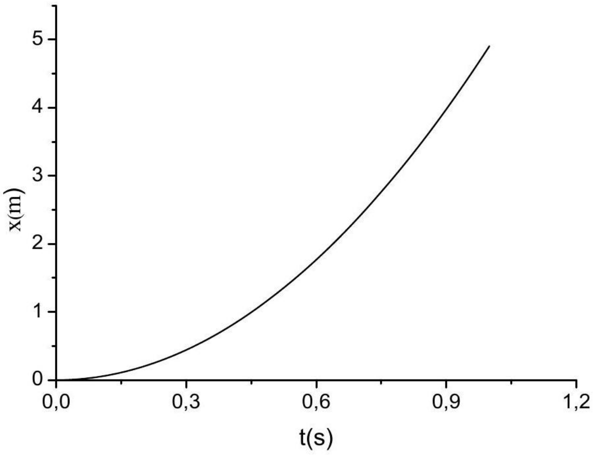


Figure 1.3. Space-Time graph

1.3.3 Displacement Vector

Let M_i and M_f be two positions of an object on the (Ox) axis at times t_i and t_f, respectively. The vector $M_i\vec{M}_f$ is called the displacement vector between t_i and t_f.

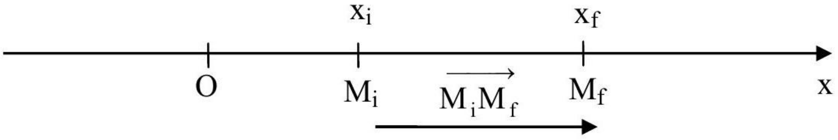


Figure 1.4

Figure II.4

According to Chasles' relation:

$$\overrightarrow{M_iM_f} = \overrightarrow{M_iO} + \overrightarrow{OM_f}$$

The relationship between this displacement vector and the position vectors is:

$$\overrightarrow{M_iM_f} = \Delta\overrightarrow{OM} = \overrightarrow{OM_f} - \overrightarrow{OM_i}$$

As a result, its component along the (Ox) axis is:

$$\Delta x = x_f - x_i$$

Note: Its magnitude should not be confused with the distance traveled. If we consider the displacement $M_i \to O \to M_f$ of an object moving on the Ox axis, $|\overrightarrow{M_iM_f}| = |x_f - x_i|$ and the distance traveled is given by $d = |\overrightarrow{M_iO}| + |\overrightarrow{OM_f}| = |x_i| + |x_f|$.

1.3.4 Velocity

Definition

Let's consider the free fall motion of a ball described by the measurements in Table II.1:

Position	M_0	M_1	M_2	M_3	M_4	M_5
t(s)	0	1	2	3	4	5
x(m)	0	5	20	45	80	125

Table II.1

Notice that for successive positions, the time interval is constant, $\Delta t = 1$ s, but the corresponding displacements are getting larger:

$$\left|\overrightarrow{M_0M_1}\right| < \left|\overrightarrow{M_1M_2}\right| < \left|\overrightarrow{M_2M_3}\right| < \left|\overrightarrow{M_3M_4}\right| < \left|\overrightarrow{M_4M_5}\right|$$

This means that the object is getting faster. To characterize this property, we introduce the notion of velocity:

The velocity vector $\vec{V}$ of a particle represents the rate of change of its position vector $\overrightarrow{OM}$ with respect to time.

- This change can involve the direction of $\overrightarrow{OM}$, its magnitude, or both.
- The unit of velocity in the International System (SI) is meters per second (m/s).

1.3.5 Average Velocity

The average velocity of an object between two instants t_i and t_f corresponding to the positions M_i and M_f is defined as:

$$\left(\vec{V}\right)_{t_i}^{t_f} = \frac{\Delta\overrightarrow{OM}}{\Delta t} = \frac{\overrightarrow{OM}_f - \overrightarrow{OM}_i}{t_f - t_i}$$

In algebraic terms, for rectilinear motion on the Ox axis:

$$(V)_{t_i}^{t_f} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

Example: Consider the free fall motion of a ball as described in Table II.1

$$(V)_{3\ s}^{5\ s} = \tfrac{x_5 - x_3}{5-3} = 40(\ m/s) \quad \left(\overrightarrow{V}\right)_{3\ s}^{5\ s} = 40\,\overrightarrow{i}\,(m/s)$$
$$(V)_{0\ s}^{5\ s} = \tfrac{x_5 - x_0}{5-0} = 25(\ m/s) \quad \left(\overrightarrow{V}\right)_{0\ s}^{5\ s} = 25\,\overrightarrow{i}\,(m/s)$$

Note 1: In the space diagram of Figure II.5

$$(V_{mx})_{t_A}^{t_B} = \frac{\Delta x}{\Delta t} = \text{the slope of the secant line (the segment) AB}$$

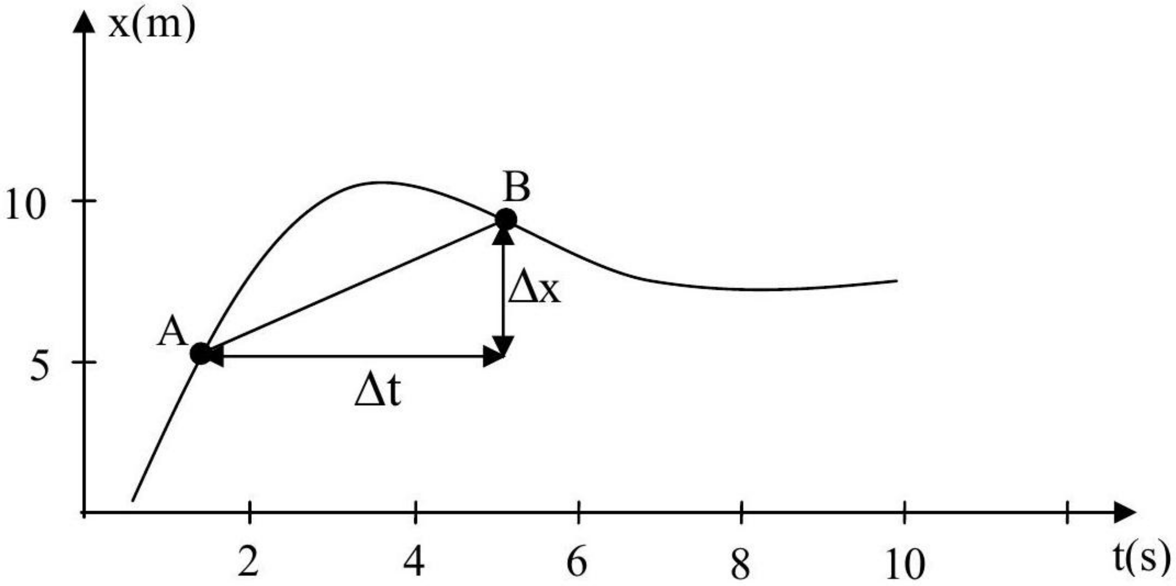


Figure 1.5. Space-Time graph

Note 2: Scalar average velocity is given by the ratio:

$$\mathrm{V} = \frac{\text{distance traveled}}{\text{time taken}} = \frac{d}{\Delta t}$$

Note 3: An average velocity $\left(\vec{V}_{\mathrm{m}}\right)_{\mathrm{t_i}}^{\mathrm{t_f}}$ characterizes the time interval $[\mathrm{t_i}, \mathrm{t_f}]$ in which it is determined. In the previous example of free fall:

$$\left(\vec{\mathrm{V}}_{\mathrm{m}}\right)_{0\ \mathrm{s}}^{5\ \mathrm{s}} \neq \left(\vec{\mathrm{V}}_{\mathrm{m}}\right)_{1\ \mathrm{s}}^{5\ \mathrm{s}} \neq \left(\vec{\mathrm{V}}_{\mathrm{m}}\right)_{2\ \mathrm{s}}^{5\ \mathrm{s}} \neq \left(\vec{\mathrm{V}}_{\mathrm{m}}\right)_{0\ \mathrm{s}}^{4\ \mathrm{s}}$$

1.3.6 Instantaneous Velocity

Definition

Sometimes, we are interested in the velocity of a particle at a particular moment t corresponding to a given position. Consider the example of the space diagram in Figure II.6. The average velocity $\left(\vec{V}_m\right)_{t_A}^{t_B}$ characterizes the interval $[t_A, t_B]$. To have a velocity that refers to the instant t_A, it is appropriate to reduce the interval $[t_A, t_B]$ to [t

This implies that the slope of the segment AB will tend towards that of the tangent to the graph at point A (see Figure II.6). Since x is the algebraic projection of the position vector on the axis, this suggests the following definition:

The instantaneous velocity vector of an object at time t is given by the relation:

$$\vec{V}(\mathrm{t}) = \lim_{\Delta \mathrm{t} \to 0} \frac{\Delta \overrightarrow{\mathrm{OM}}}{\Delta \mathrm{t}} = \frac{\mathrm{d}\overrightarrow{\mathrm{OM}}}{\mathrm{dt}}$$

Algebraically, for rectilinear motion along (Ox):

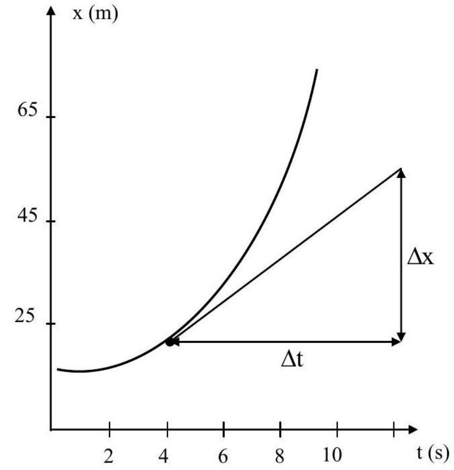
$$\mathrm{V_x(t)} = \frac{\mathrm{dx}}{\mathrm{dt}} = \text{the slope of the tangent to the space diagram}$$
$$\text{at the point corresponding to time t.}$$

The graph of $\mathrm{V_x(t)}$ is called the velocity diagram.

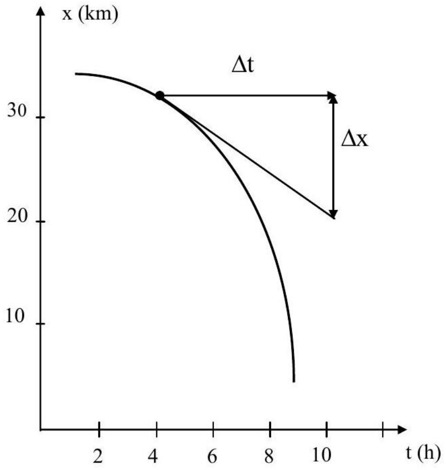
1.3.7 Measuring Velocity

First method: if we have the space diagram, we can obtain the algebraic values, V_x, by measuring the slopes of the tangents to the graph at the points considered. It is important to pay attention to units and signs.

Example



$V_x(m/s) > 0$



$V_X(km/h) < 0$

Second method: the average velocity can be confused with instantaneous velocity in two cases:

1st case: when the velocity is constant (uniform motion) and we have:

$$\overrightarrow{V}(t) = \left(\overrightarrow{V}\right)_{t_i}^{t_f} \text{ for all } t, t_i \text{ and } t_f$$

2nd case: when the time interval $\Delta t = t_f - t_i$ is small enough, we can equate the average velocity $\left(\vec{V}\right)_{t_i}^{t_f}$ to the instantaneous velocity $\vec{V}(t)$ at the midpoint of the interval: $[t_i, t_f]$

$$\Delta t = t_f - t_i < \varepsilon \text{ (small)} \Rightarrow \overrightarrow{V}(t) = \left(\overrightarrow{V}\right)_{t_i}^{t_f} \text{ with } t = \frac{t_i + t_f}{2}$$

1.4 Acceleration

Velocity may vary with time, and we characterize this change by introducing the concept of acceleration. The acceleration vector $\vec{a}$ represents the rate of change of the velocity vector $\vec{V}$ with respect to time.

- This variation can affect the direction of velocity, its magnitude, or both.
- The unit of acceleration in the International System of Units (SI) is meters per second squared (m/s^2).

Definitions

By following the approach used in introducing velocity, we can define the following quantities:

a. The average acceleration

The average acceleration of a mobile between two instants t_i and t_f is given by the ratio:

$$\vec{a}|_{t_i}^{t_f} = \frac{\Delta\vec{V}}{\Delta t} = \frac{\vec{V}_f - \vec{V}_i}{t_f - t_i}$$

In algebraic terms, and in the case of rectilinear motion along the Ox axis,

$$a|_{t_i}^{t_f} = \frac{\Delta V_x}{\Delta t} = \frac{V_{x_f} - V_{x_i}}{t_f - t_i} \tag{1.1}$$

b. Instantaneous Acceleration:

Average acceleration characterizes the time interval $[t_i, t_f]$. The transition to the instantaneous quantity is similar to that of velocity. Acceleration at a given time t is defined as:

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Acceleration at an instant t is obtained by taking the limit as the time interval Δt approaches zero. It represents the rate of change of velocity at that specific moment.

Algebraically, in the case of rectilinear motion along the Ox axis:

$$a_x(t) = \frac{dV_x}{dt}$$

where $a_x(t)$ represents the acceleration, and $\frac{dV_x}{dt}$ is the slope of the tangent to the velocity-time graph. Note: It should be noted that

$$a_x(t) = \frac{d^2x(t)}{dt^2} = \frac{dV_x(t)}{dt} = \frac{d^2}{dt^2}(x(t))$$

Thus, the orientation of the concavity of the position-time graph indicates the sign of a_x:

- $a_x > 0$: Concavity is directed towards the positive direction of the Ox axis.
- $a_x < 0$: Concavity is directed towards the negative direction of the Ox axis.

Example: For the motion described by the position-time graph

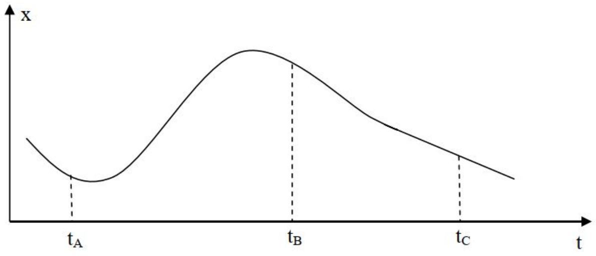


Figure 1.6. Space-Time graph

$$a_x(t_A) > 0, \quad a_x(t_B) < 0, \quad a_x(t_c) = 0$$

C. Measurement of Acceleration

Just as with velocity, acceleration can be determined:

- by measuring the slopes of the tangents to the velocity-time graphs;

- by approximating the average acceleration

$$\vec{a}_{\text{avg}}|_{t_i}^{t_f} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

with the instantaneous acceleration

$\vec{a}(t)$ in the middle of the time interval $[t_i, t_f]$ if it is small enough:

$$|t_f - t_i| \text{ small enough } \Rightarrow \vec{a}(t) = \vec{a}_{\text{avg}}|_{t_i}^{t_f} \quad \textbf{at the moment } t = \frac{t_i + t_f}{2}$$

1.5 Integral Relationships

In the preceding section, it became evident that one can transition from position to velocity and from velocity to acceleration through differentiation. In this part, we will demonstrate that the reverse transition can be accomplished through integration.

1.5.1 Passage from Velocity to Position

Position and velocity are related by the equation:

$$V_x(t) = \frac{dx}{dt}$$

In other words:

$$dx = V_x(t)dt$$

By integrating between two moments t_i and t_f, we obtain:

$$x(t_f) - x(t_i) = \int_{t_i}^{t_f} V(t)\,dt$$

As a consequence, if we know the position x_0 of an object at a specific time t_0 and have the expression for its velocity as a function of time, we can determine its position $x(t)$ at any given time t by writing:

$$x(t) = x_0 + \int_{t_0}^{t} V(t)\,dt$$

This allows us to find the position of the object at any time t given its initial position and velocity profile.

Graphical Equivalence of Integrals and Areas

Graphically, the equivalence between the value of the integral of a function and the area located between the curve of its graph and the x-axis allows us to write:

$$x(t_f) - x(t_i) = \int_{t_i}^{t_f} V(t)\,dt = \text{area enclosed by the curve of } V(t), \text{ the time axis, and the lines } t = t_i \text{ and } t = t_f \quad \text{(II.20)}$$

The areas being algebraic, any portion located below the (Ot) axis is negative, and the one above it is positive.

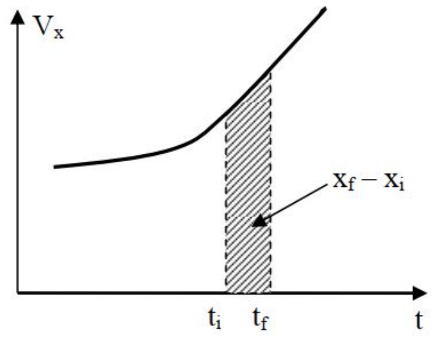


Figure 1.7. Velocity-time graph.

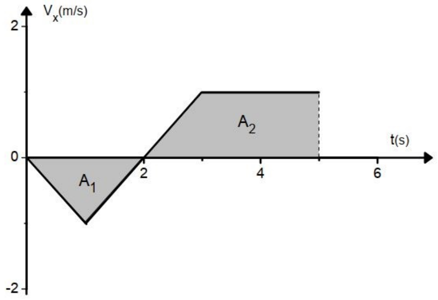


Figure 1.8. Velocity-time graph.

In the previous example: 1. For $0s \leq t \leq 2s$, the area A_1 is negative:

$$A_1 = \int_0^2 V(t)\,dt = -1\,\mathrm{m}$$

2. For $2s \leq t \leq 5s$, the area A_2 is positive:

$$A_2 = \int_2^5 V(t)\,dt = 2.5\,\mathrm{m}$$

distance traveled
To calculate the distance traveled between $0s$ and $5s$, we sum the absolute values of these areas:

$$\Delta x_{0s}^{5s} = |A_1| + |A_2| = 1\,\mathrm{m} + 2.5\,\mathrm{m} = 3.5\,\mathrm{m}$$

1.5.2 Transition from Acceleration to Velocity

Acceleration and velocity are related by:

$$a_x(t) = \frac{dV_x}{dt}$$

or:

$$dV_x = a_x(t)dt$$

The integration between two instants t_i and t_f yields:

$$V_x(t_f) - V_x(t_i) = \int_{t_i}^{t_f} a_x(t)\,dt$$

This can be expressed as:

$$V_x(t) = Vx_0 + \int_{t_0}^{t} a_x(t)\, dt$$

Graphical Equivalence of Integrals and Areas

$$\Delta V_x|_{t_i}^{t_f} = V_x(t_f) - V_x(t_i) = \int_{t_i}^{t_f} a_x(t)\, dt$$

This expression represents the algebraic area enclosed by the curve of $a(t)$, the time axis, and the lines $t = t_i$ and $t = t_f$.

This variation is algebraic, it can be positive as well as negative.

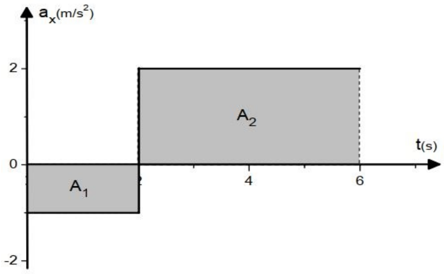


Figure 1.9. Velocity-time graph.

Kinematic Study of Special Rectilinear Motions

a. Uniform Rectilinear Motion

We consider the motion of an object to be uniform when the algebraic value of its velocity is constant. It is motion without acceleration in accordance with the relation:

$$a_x(t) = \frac{dV_x}{dt}$$

From the integral formula, for $x_0 = x(0)$, we obtain: $\mathrm{x(t)} = \mathrm{x}_0 + \int_{t_0}^{t} V(t)\, dt = V_x(t).t + x_0 Diagrams$:

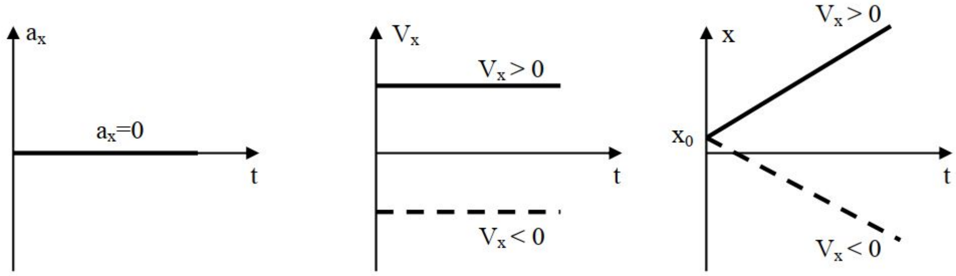


Figure 1.10. Velocity-time graph.

b. Uniformly Varied Rectilinear Motion

It is said that motion is uniformly varied when the acceleration of the mobile is constant. By using integral calculus, we obtain

$$V_x(t) = Vx_0 + \int_{t_0}^{t} a_x(t)\, dt = a_x t + Vx_0, \quad \text{with} \quad Vx_0 = Vx(0)$$

For the position:

$$x(t) = x_0 + \int_{t_0}^{t} (a_x t + V x_0)\, dt = \frac{1}{2} a t^2 + V x_0 t + x_0, \quad \text{with} \quad x_0 = x(0)$$

Examples of diagrams:

- Case where $a_x > 0$ (figure 5)

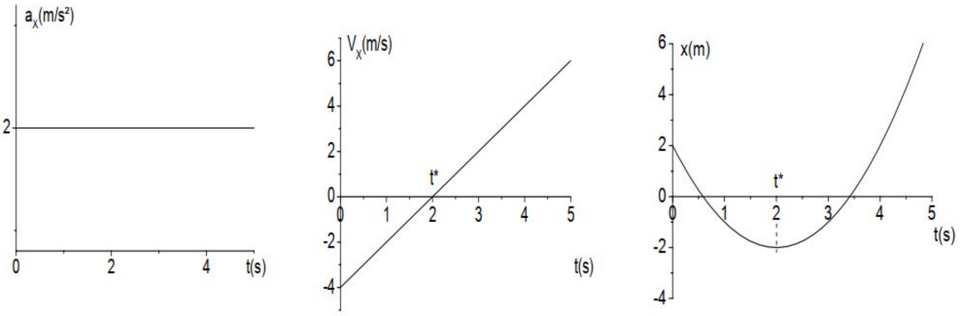


Figure 1.11. Velocity-time graph.

- Case where $a_x < 0$ (figure 6)

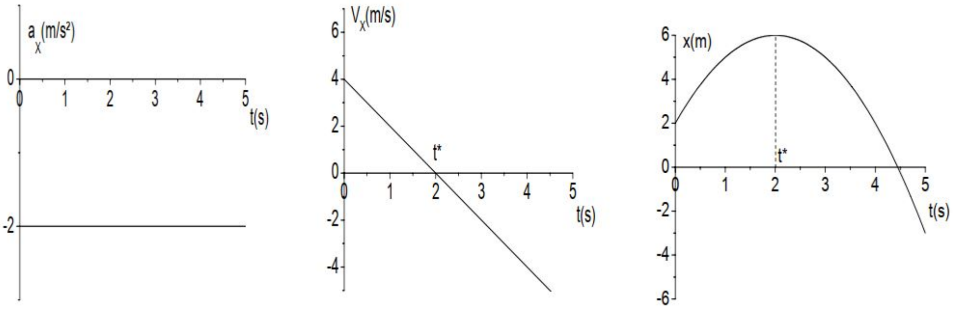


Figure 1.12. Acceleration-time graph.

Time-independent formula

This formula is valid only for uniformly varied motion (constant acceleration) $a_x = constant$.

$$a_x(t) = \frac{dV_x}{dt} \Rightarrow dV_x = a_x dt$$

By multiplying both sides by V_x, we obtain: $V_x\, dV_x = V_x\, a_x\, dt$ as $V_x = \frac{dx}{dt} \Rightarrow V_x dt = dx$, So,

$$V_x dV_x = a_x dx$$

Let :

$$\int_{V_{x1}}^{V_{x2}} V_x dV_x = \int_{x_1}^{x_2} axdx$$

as a_x constant, we obtain :

$$V_{x2}^{2} - V_{x1}^{2} = 2a_x(x_2 - x_1) \tag{1.2}$$

1.5.3 Particular Natures of Rectilinear Motion

- Constant $V_x \Rightarrow$ the motion is uniform rectilinear.

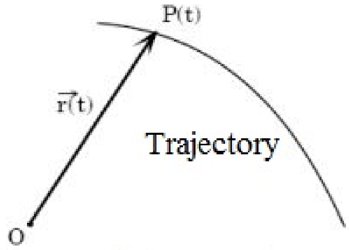
- Constant $a_x \Rightarrow$ the motion is uniformly varied rectilinear.
- $a_x V_x < 0$: the motion is rectilinearly decelerated (uniformly if a_x is constant).
- $a_x V_x > 0$: The motion is rectilinear decelerated or retarded. (uniformly if a_x is constant).

1.6 Movement in Space

This is a movement with a trajectory (path) that is an arbitrary curve, meaning it is not necessarily a straight line.

1.6.1 Positioning

To describe the motion of an object, it's necessary to select an origin O that will serve as a reference point. Its position $P(t)$ is identified at each moment by the position vector $\vec{r}(t)$, which is $\overrightarrow{OP(t)}$ (see Figure 3.1). To analyze the motion, a coordinate system related to the origin O must be defined. The choice of this system depends on the specific properties of the problem being considered.



To simplify the study of motion in space, we will, in the first instance, use the Cartesian coordinate system that is familiar to us. We will introduce others in this chapter.

The adopted Cartesian coordinate system consists of three axes (Ox, Oy, Oz) equipped with unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ as shown in Figure (3.2). The elements ($\vec{i}$, $\vec{j}$, $\vec{k}$) form an orthonormal basis (they are mutually perpendicular and have magnitudes equal to one).

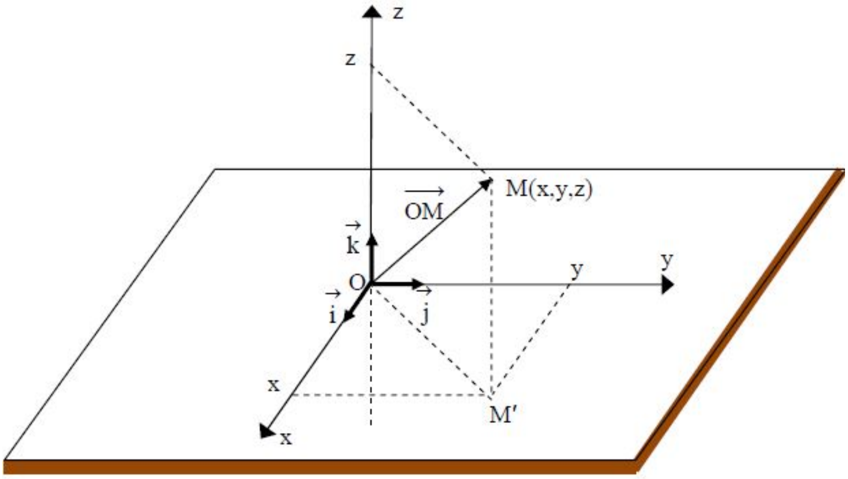


Figure 1.13

In this reference frame, the position vector of a mobile point M is written as follows:

$$\vec{r} = \overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

The relations that represent the coordinates of point M in our reference frame as functions of time are given by:

$$x = x(t), \quad y \quad = y(t), \quad z = z(t).$$

They represent the parametric equations of motion

Example :

Figure ?? depicts the trajectory of a helical motion. At each moment, the position M of the mobile is identified by its Cartesian coordinates, which are described by the parametric equations:

$$x = x_0\, cos(\omega t), \quad y \ = x_0\, sin(\omega t), \quad z = V_z(t).$$

where x_0 and V_z are constants

Remarks:

- In the general case, we locate a position using its three coordinates in a three-axis system (three-dimensional).
- When the motion takes place in a plane (planar motion), we can reduce the reference frame to a two-dimensional system, for example, with axes (Ox, Oy) contained within the plane of motion.

- The equation of the trajectory is obtained by eliminating the variable t from the parametric equations. For example, in the case of the motion of a projectile launched from the origin O with an initial velocity $\overrightarrow{V_0}$, horizontal, the parametric equations are as follows:

$$x = V_0(t), \quad y \ = -\frac{1}{2}gt^2$$

For an ascending (Oy) axis, we can then write: $t = \frac{x}{V_0}$. By substituting this expression into the equation for y, we obtain the equation for the trajectory:

$$y = -\frac{1}{2}\frac{g}{V_0}x^2$$

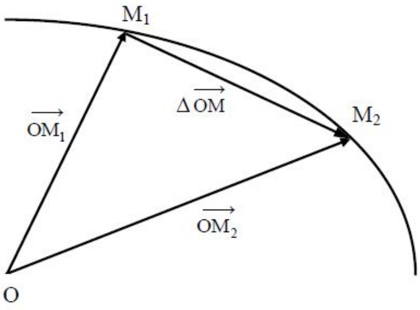
When the parametric equations contain trigonometric functions of time, it's important to try to proceed by exploiting certain relationships that characterize them. For example, if

$$x(t) = cos(t)^2, \quad y(t) \ = sin(t)^2,$$

So, we obtain

$$x^2 + y^2 = \cos(t)^2 + \sin(t)^2 = 1$$

1.6.2 Displacement Vector



If at time t_1, a mobile is located at point M_1 as follows:

$$O\vec{M}_1(t_1) = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

At time t_2, it is located at M_2 as follows:

$$O\vec{M}_2(t_1) = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

The displacement vector is the vector $\overrightarrow{M_1M_2}$.

Its relation with the position vectors is as follows:

$$\overrightarrow{M_1M_2} = \Delta\overrightarrow{OM} = \Delta\vec{r} = \overrightarrow{OM_2} - \overrightarrow{OM_1}$$

In the Cartesian coordinate system:

$$\Delta\vec{r} = \Delta\overrightarrow{OM} = \Delta x\vec{i} + \Delta y\vec{j} + \Delta z\vec{k}$$

with $\Delta x = (x_2 - x_1), \quad \Delta y = (y_2 - y_1), and \quad \Delta z(z_2 - z_1)$

1.7 Velocity Vector

1.7.1 Average Velocity Vector ($\vec{V}|_{t_i}^{t_f}$)

Let M_1 be the position of the mobile at time t_1 and M_2 be the position at time t_2. In this case, we define the average velocity vector between these two instants as:

$$\vec{V}|_{t_i}^{t_f} = \frac{\overrightarrow{M_1M_2}}{\Delta t} = \frac{\Delta\overrightarrow{OM}}{\Delta t}$$

where $M\vec{M}_2$ is the displacement vector between M_1 and M_2.

- Its magnitude is given by:

$$\|\vec{V}\|_{t_i}^{t_f} = \left\|\frac{\overrightarrow{M_1M_2}}{\Delta t}\right\| = \left\|\frac{\Delta\overrightarrow{OM}}{\Delta t}\right\|$$

- It has the same direction and sense as $\overrightarrow{M_1M_2}$.
- It is a sliding vector; its point of application is a point on the segment $[M_1, M_2]$.

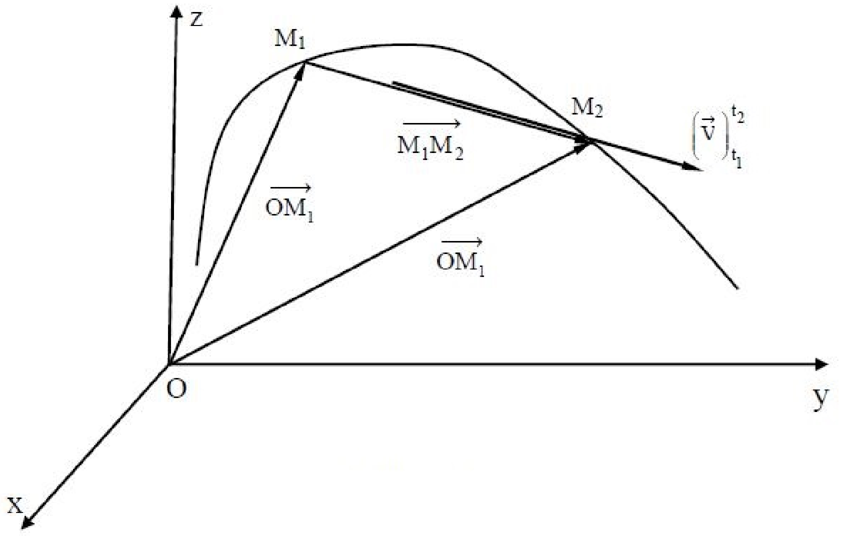


Figure 1.14

Its analytical expression in the Cartesian coordinate system is:

$$\vec{V}|_{t_i}^{t_f} = \frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j} + \frac{\Delta z}{\Delta t}\vec{k} \tag{1.3}$$

$$= V_x\vec{i} + V_y\vec{j} + V_z\vec{k} \tag{1.4}$$

1.7.2 Instantaneous Velocity $\mathbf{V}(t)$

Similar to rectilinear motion, instantaneous velocity, in its general sense, provides more precise information than the average velocity vector. It defines the velocity of the mobile at each instant.

Instantaneous velocity is also obtained by reducing the time interval Δt to zero, starting from the average velocity. Thus, instantaneous velocity $\vec{V}_1(t_1)$ is obtained by considering the limit of $\vec{V}|_{t_i}^{t_f}$ As M_2 approaches M_1, graphically, the direction of the displacement vector tends toward that of the tangent at M_1. Mathematically, this is expressed as:

$$\vec{V}_1(t_1) = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t} = \frac{d\overrightarrow{OM}}{dt}$$

Characteristics:

- The instantaneous velocity vector is, at every instant, tangent to the trajectory.
- Its direction is along the direction of motion.

Components in a Cartesian Coordinate System:
In the basis $(\mathbf{O}, \tilde{\mathbf{i}}, \tilde{\mathbf{j}}, \tilde{\mathbf{k}})$, the instantaneous velocity vector $\vec{V}$ is expressed as:

$$\vec{V} = \frac{d}{dt}\vec{OM} = \frac{d}{dt}(x\,\vec{i} + y\vec{j} + z\,\vec{k}) = \frac{dx}{dt}\,\vec{i} + \frac{dy}{dt}\,\vec{j} + \frac{dz}{dt}\,\vec{k}$$

You can also represent it as:

$$\vec{V} = V_x\,\vec{i} + V_y\,\vec{j} + V_z\,\vec{k}$$

where:
$\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are the time derivatives of the position coordinates.
V_x, V_y, V_z are the components of $\vec{V}$ along the coordinate axes. Let us have

$$\begin{cases} V_x = \dfrac{dx}{dt} = & \text{Slope of the tangent to the graph of } x(t) \\ V_y = \dfrac{dy}{dt} = & \text{Slope of the tangent to the graph of } y(t) \\ V_z = \dfrac{dz}{dt} = & \text{Slope of the tangent to the graph of } z(t) \end{cases}$$

with magnitude

$$\|\vec{V}\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Planar Motion:

The basis is reduced to base $(\vec{i}\,,\,\vec{j})$, and the velocity vector $\overrightarrow{V}(t)$ is expressed as:

$$\vec{V}(t) = V_x\,\vec{i} + V_y\,\vec{j}$$

where $\vec{i}$ and $\vec{j}$ form the basis for the planar motion, and V_x and V_y are the respective components along these directions. In this case we have :

$$\overrightarrow{V}(t) = \begin{cases} V_x = \dfrac{dx}{dt} \\ V_y = \dfrac{dy}{dt} \end{cases} \qquad \|\vec{V}\| = \sqrt{V_x^2 + V_y^2}$$

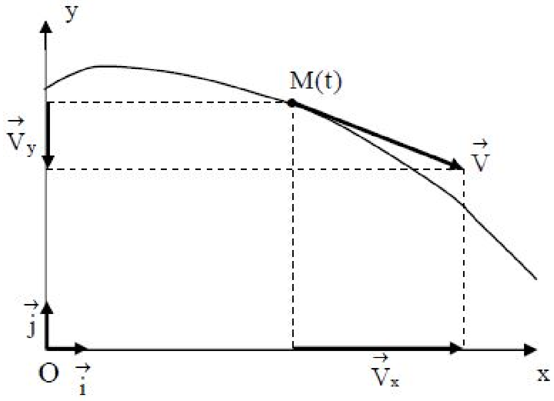


Figure 1.15

1.8 Acceleration Vector

1.8.1 Average Acceleration Vector ($\vec{a}|_{t_i}^{t_f}$)

The relative change in velocity during the time interval $\Delta t = t_2 - t_1$ is given by the average acceleration vector:

$$\vec{a}|_{t_i}^{t_f} = \frac{\Delta\vec{V}}{\Delta t} = \frac{\vec{V}_2 - \vec{V}_1}{t_f - t_i}$$

when its length is given by :

$$\parallel \vec{a}|_{t_i}^{t_f} \parallel = \frac{\|\Delta\vec{V}\|}{\Delta t} \tag{1.5}$$

• It has the same direction and orientation as the change in velocity vector $\Delta\vec{V}$.

• Generally $\vec{a}|_{t_i}^{t_f}$, it is applied to the point M where the mobile is located at time t. The midpoint of the interval is given by $t = \frac{t_1+t_2}{2}$.

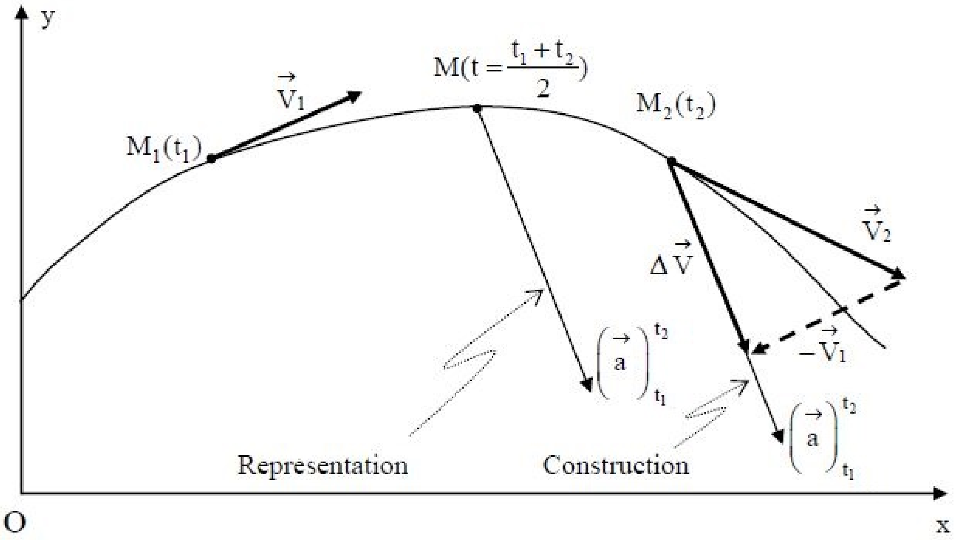


Figure 1.16

From an algebraic perspective, this leads us to write in the Cartesian coordinate system:

$$\vec{a}|_{t_i}^{t_f} = \frac{\Delta V_x}{\Delta t}\,\vec{i} + \frac{\Delta V_y}{\Delta t}\,\vec{j} + \frac{\Delta V_z}{\Delta t}\,\vec{k} \tag{1.6}$$

$$= a_x\,\vec{i} \quad + a_y\,\vec{j} \quad + a_z\,\vec{k} \tag{1.7}$$

1.8.2 Instantaneous Acceleration Vector $\vec{a}(t)$

As before, we will take the limit as $\Delta t \to 0$ to obtain the instantaneous acceleration.

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{V}}{\Delta t} = \frac{d\vec{V}}{dt}$$

The magnitude, direction, and sense of acceleration can typically only be specified by introducing its components in a reference system. However, $\vec{a}$ is always oriented towards the concave side of the trajectory.

In Cartesian coordinate systems, a vector $\vec{a}$ is typically represented as:

$$\begin{aligned}\vec{a}(t) &= \frac{d}{dt}\left(V_x\vec{i} + V_y\vec{j} + V_z\vec{k}\right)\\ &= \frac{dV_x}{dt}\vec{i} + \frac{dV_y}{dt}\vec{j} + \frac{dV_z}{dt}\vec{k}\\ &= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\end{aligned}$$

with

$$\vec{a}(t)\begin{cases} a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt^2} = \text{slope of the tangent to the graph of } V_x(t)\\ a_y = \frac{dV_y}{dt} = \frac{d^2y}{dt^2} = \text{slope of the tangent to the graph of } V_y(t)\\ a_z = \frac{dV_z}{dt} = \frac{d^2z}{dt^2} = \text{slope of the tangent to the graph of } V_z(t)\end{cases}$$

and

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

1.9 Transition from Acceleration to Velocity and Position

Recall that $\vec{V}(t)$ and $\vec{a}(t)$ can be obtained by differentiation from $\vec{r}(t)$:

$$\vec{V}(t) = \frac{d\,\vec{r}(t)}{dt} \quad \text{and} \quad \vec{a}(t) = \frac{d\,\vec{V}(t)}{dt}$$

Sometimes, only the acceleration is known, and we need to find the velocity and then the position by integration. To do this, we can write the previous relations in the form:

$$d\,\vec{r}(t) = \vec{V}(t)\,dt \quad \text{and} \quad d\,\vec{V}(t) = \vec{a}(t)\,dt$$

Algebraically, these relations give:

$$\begin{aligned}dx(t) &= V_x(t)dt, \quad dy(t) = V_y(t)dt, \quad dz(t) = V_z(t)dt\\ dV_x(t) &= a_x(t)dt, \quad dV_y(t) = a_y(t)dt, \quad dV_z(t) = a_z(t)dt\end{aligned}$$

The integration of these equations leads to the following:

- If we know the position vector $\vec{OM}_0(x_0, y_0, z_0)$ of a particle at a particular time t_0, and the time-dependent expression of its velocity $\vec{V}(t)(V_x(t), V_y(t), V_z(t))$, we can determine its position at any time t using:

$$\begin{aligned}x(t) &= x_0 + \int_{t_0}^{t} V_x(t)dt\\ y(t) &= y_0 + \int_{t_0}^{t} V_y(t)dt\\ z(t) &= z_0 + \int_{t_0}^{t} V_z(t)dt\end{aligned}$$

- If we know the velocity vector $\vec{V_0}(V_{x0}, V_{y0}, V_{z0})$ of a particle at a particular time t_0, and the time-dependent expression of its acceleration $\vec{a}(t)(a_x(t), a_y(t), a_z(t))$, we can determine its velocity at any time t using:

$$V_x(t) = V_{x0} + \int_{t_0}^{t} a_x(t)dt$$
$$V_y(t) = V_{y0} + \int_{t_0}^{t} a_y(t)dt$$
$$V_z(t) = V_{z0} + \int_{t_0}^{t} a_z(t)dt$$

1.9.1 Approximation of Instantaneous Quantities Using Averages

Just like in rectilinear motion, if $\Delta t = t_f - t_i$ is small enough, we can equate:

- The average velocity $\left(\vec{V}_m\right)_{t_i}^{t_f}$ with the instantaneous velocity $\vec{V}_m(t)$ at the midpoint of the time interval $[t_i, t_f]$;
- The average acceleration $(\vec{a}_m)_{t_i}^{t_f}$ with the instantaneous acceleration $\vec{a}(t)$ at the midpoint of the time interval $[t_i, t_f]$.

$$\Delta t = t_f - t_i < \varepsilon \text{ small}, \Rightarrow \begin{cases} \vec{V}(t) = \vec{V}\big|_{t_i}^{t_f} \\ \vec{a}(t) = \vec{a}\big|_{t_i}^{t_f} \end{cases} \quad \text{at time} \quad t = \frac{t_i + t_f}{2}$$

1.10 Curvilinear Abscissa, Velocity, and Acceleration

If the trajectory of a mobile M is known, we can:
• Orient it in an arbitrary direction;
• Choose a fixed reference point M_0 on this trajectory;
• Choose a graphic unit.
The algebraic value of the arc $(\overrightarrow{M_0M})$ is denoted as the **curvilinear abscissa** s of point M.

Example: On a road map, distances are determined based on curvilinear abscissas. The origin is the kilometer zero point, and the unit is the kilometer.

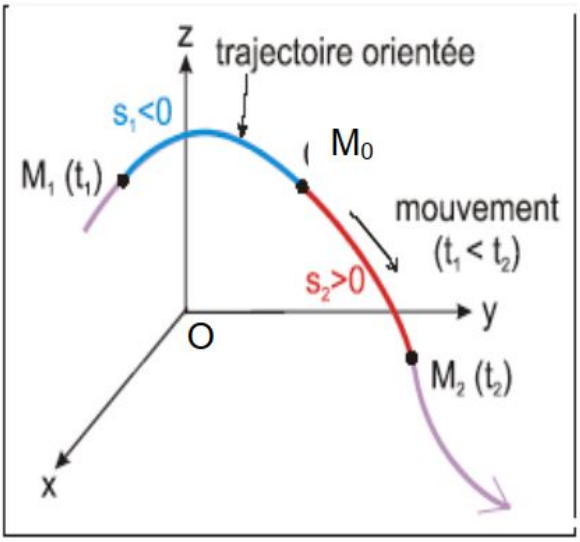


Figure 1.17

Respectively, we define curvilinear velocity and acceleration using the following equations:

$$V(t) = \frac{ds(t)}{dt}, \qquad a(t) = \frac{dV(t)}{dt}$$

Example: Varied circular motion on a trajectory with radius R we have

$$s = R\theta, \quad V = \frac{ds}{dt} = R\,\frac{d\theta}{dt}, \quad \text{and} \quad a = R\,\frac{d^2\theta}{dt^2}$$

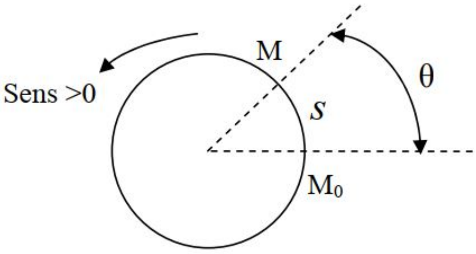


Figure 1.18

Figure II.24

1.11 Intrinsic Components of Acceleration

1.11.1 Definitions

In some cases, to determine the acceleration at a point M, we use its intrinsic components, which are its algebraic projections (figure 13)

- a_t on a tangential axis (MT) with the unit vector $\vec{u}_t$, directed in the direction of motion.
- a_n on a normal axis (MN) with the unit vector $\vec{u}_n$, oriented towards the concave side of the trajectory.

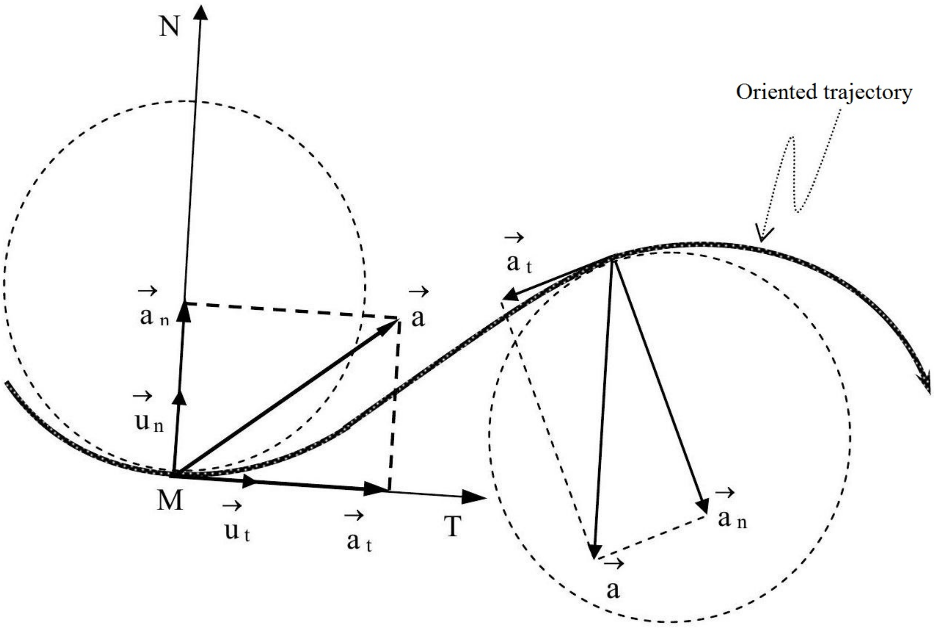


Figure 1.19

Hence the expression

$$\overrightarrow{a} = \overrightarrow{a}_{\mathrm{t}} + \overrightarrow{a}_{\mathrm{n}}$$
$$\overrightarrow{a}_{\mathrm{t}} + \overrightarrow{u}_{\mathrm{n}}\overrightarrow{u}_{\mathrm{n}}$$

a_{t} and a_{n} are, respectively, the tangential and normal components of acceleration.

Note: The unit vectors $\overrightarrow{\mathrm{u}}_{\mathrm{t}}$ and $\overrightarrow{\mathrm{u}}_{\mathrm{n}}$ form an orthonormal basis called the Frenet basis. It is a projection base (or frame) linked to the position of the mobile point M. In physics, do not confuse this concept with that of a reference frame, which is linked to an observer.

1.11.2 Expressions of the Tangential and Normal Components of Acceleration

Since the velocity vector is tangential, it is expressed in the Frenet frame as:

$$\overrightarrow{\mathrm{V}} = \mathrm{V}\overrightarrow{\mathrm{u}}_{\mathrm{t}}$$

Where V is the magnitude. Let us differentiate this expression with respect to time to find the acceleration:

$$\begin{aligned} \overrightarrow{a} &= \frac{d\vec{V}}{dt} \\ \overrightarrow{\mathrm{a}} &= \frac{dV}{dt}\overrightarrow{\mathrm{u}}_{\mathrm{t}} + \mathrm{V}\frac{d\,\vec{\mathrm{u}}_t}{dt} \end{aligned}$$

Note that

$$\frac{d\overrightarrow{\mathrm{u}}_{\mathrm{t}}}{dt} = \frac{d\,\vec{\mathrm{u}}_t}{ds}\frac{ds}{dt}$$

Recalling that

$$\frac{ds}{dt} = \mathrm{V}$$

The vectors of the Frenet basis continuously form an orthonormal basis, and their derivatives satisfy certain relations. In particular, we assume:

$$\frac{d\vec{u}_t}{ds} = \frac{1}{\rho}\vec{u}_n$$

Here, $\rho(s)$ is called the radius of curvature of the trajectory at the considered point. If the trajectory is sufficiently smooth, there is always one and only one circle that is tangent to it; then, ρ is its radius.

This leads to the following explicit expression for acceleration:

$$\begin{aligned} \vec{a} &= \frac{dV}{dt}\vec{u}_t + \frac{V^2}{\rho}\vec{u}_n \\ &= a_t\vec{u}_t + a_n\vec{u}_n \end{aligned}$$

Interpretation:

- $a_t = \frac{dV}{dt}$ indicates that the **tangential component** is related to **changes in the velocity magnitude**, i.e., if the object is moving faster or slower.
- $a_n = \frac{V^2}{\rho}$ with the presence of ρ indicating that the **normal component** signifies that the trajectory is curved. As a result, **the direction of the velocity vector varies**.

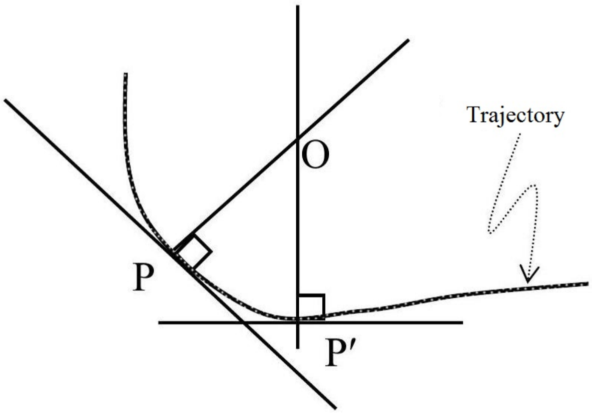
Examples:

- Varied rectilinear motion: **"rectilinear"** means there is no change in the direction of the velocity vector. In this case, **the radius of curvature** ρ of the trajectory is **infinite**, and thus, $a_n = 0$. So, "Varied" means $a_t \neq 0$."

- Uniform **circular** motion: "circular" means that the object moves along a circular trajectory with a radius $\mathrm{R} = \rho$. As a result, $\mathrm{a_n} = \frac{\mathrm{V}^2}{\mathrm{R}} \neq 0$. **"Uniform"** means $\mathrm{a_t} = 0$.

1.11.3 Additional Information:

α) The radius of curvature at a point P on the trajectory



Let P′ be a neighboring point of P and O be the point of intersection of the normals to the trajectory at P and P′. The radius of curvature, ρ, is equal to the limit of the distance (OP) as P′ approaches P.

β Nature of Motion

- $\rho =$ constant: the motion is circular.
- $\mathrm{a_t} = 0$: the motion is uniform.
- $\mathrm{a_t} =$ constant: the motion is uniformly varied.
- $a_t > 0$: the motion is accelerated (uniformly if $a_t =$ constant).
- $a_t < 0$: the motion is decelerated or retarded (uniformly if $a_t =$ constant).

Exercise:

Show that the motion described by the parametric equations

$$\begin{cases} \mathrm{x(t)} = \cos \mathrm{t}^2 (\mathrm{m}) \\ \mathrm{y(t)} = \sin \mathrm{t}^2 (\mathrm{m}) \end{cases}$$

in the Cartesian coordinate system $(\mathrm{O, x, y})$ is uniformly accelerated circular motion.

Answer:

Note that $\mathrm{x}^2 + \mathrm{y}^2 = \left(\cos \mathrm{t}^2\right)^2 + \left(\sin \mathrm{t}^2\right)^2 = 1$. The trajectory is then a circle centered at O with a radius of $\mathrm{R} = 1$ m.

The velocity vector has components

$$\vec{V} \begin{cases} V_x = \frac{dx}{dt} = -2t \sin t^2 (\mathrm{m/s}) \\ V_y = \frac{dy}{dt} = 2t \cos t^2 (\mathrm{m/s}) \end{cases}$$

and a magnitude of

$$|\vec{V}| = \sqrt{V_x^2 + V_y^2} = \sqrt{4t^2 \left(\left(\cos t^2\right)^2 + \left(\sin t^2\right)^2 \right)} = 2t \quad (m/s)$$

The intrinsic components of acceleration are:

$$\rightarrow \begin{cases} \mathrm{a_t} = \frac{dV}{dt} = 2(\mathrm{m/s^2}) \\ \mathrm{a_n} = \frac{V^2}{\mathrm{R}} = 4t^2(\mathrm{m/s^2}) \end{cases}$$

Finally, the tangential component of acceleration is constant and positive, indicating uniformly accelerated motion.

1.12 Study of Motion in Polar Coordinates

1.12.1 Definition

This coordinate system is suitable for studying planar motions with rotational symmetry. The reference is made relative to a polar axis (Ox), with origin O called the pole. We can then locate the position of any point M in the plane containing (Ox) by:

- The polar radius $\mathrm{r}(t) = |\overrightarrow{\mathrm{OM}}(t)|$
- The polar angle $\theta(t) = (\overrightarrow{\mathrm{Ox}}, \overrightarrow{\mathrm{OM}})$

which can vary with time. Note that the radius r and the angle θ (as defined in Figure 20) are positive.

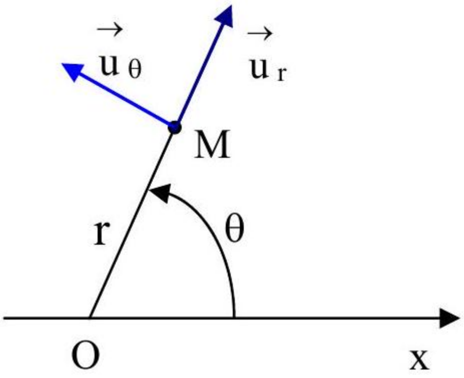


Figure 1.20

1.12.2 The Basis

In this system, we use the basis consisting of the unit vectors:

- $\vec{\mathrm{u}}_\mathrm{r}$ with the direction and sense of $\overrightarrow{\mathrm{OM}}$;
- $\vec{\mathrm{u}}_\theta$ obtained by rotating $\vec{\mathrm{u}}_\mathrm{r}$ by an angle $\pi/2$ counterclockwise.

The basis $(\vec{u}_r, \vec{u}_\theta)$ is linked to point M, and therefore the directions of the unit vectors may vary with time. Their derivatives satisfy a number of relationships, including:

$$\frac{\mathrm{d}\vec{\mathrm{u}}_\mathrm{r}}{\mathrm{dt}} = \frac{\mathrm{d}\theta}{\mathrm{dt}}\vec{\mathrm{u}}_\theta \quad ; \quad \frac{\mathrm{d}\vec{\mathrm{u}}_\theta}{\mathrm{dt}} = -\frac{\mathrm{d}\theta}{\mathrm{dt}}\vec{\mathrm{u}}_\mathrm{r}$$

1.12.3 Relationship with Cartesian Coordinates

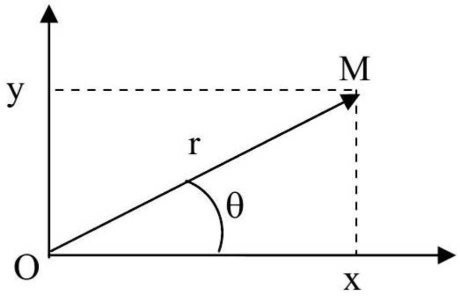


Figure 1.21

The polar coordinates r and θ of point M are related to Cartesian coordinates by the following equations:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

1.12.4 Position, Velocity, and Acceleration Vectors

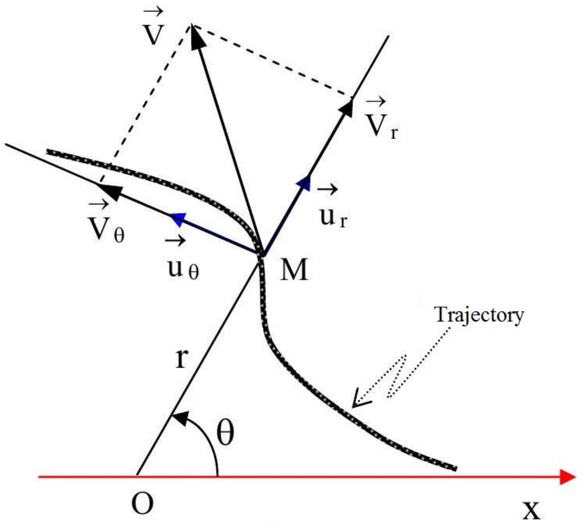
Position Vector:

The definitions of $\mathrm{r}(t)$ and $\overrightarrow{\mathrm{u}}_\mathrm{r}$ allow us to write:

$$\overrightarrow{\mathrm{OM}} = \mathrm{r}\overrightarrow{\mathrm{u}}_\mathrm{r}$$

Velocity Vector:

By definition:



$$\begin{aligned}
\overrightarrow{\mathrm{V}} &= \frac{\mathrm{d}\overrightarrow{\mathrm{OM}}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\mathrm{r}(t)\overrightarrow{\mathrm{u}}_\mathrm{r}(t)\right) \\
&= \frac{\mathrm{dr}}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\mathrm{r} + \mathrm{r}\frac{\mathrm{d}\overrightarrow{\mathrm{u}}_\mathrm{r}}{\mathrm{dt}} \\
&= \frac{\mathrm{dr}}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\mathrm{r} + \mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\theta \\
&= \mathrm{V}_\mathrm{r}\overrightarrow{\mathrm{u}}_\mathrm{r} + \mathrm{V}_\theta\overrightarrow{\mathrm{u}}_\theta
\end{aligned}$$

We can identify the velocity components as follows:

$$\overrightarrow{\mathrm{V}}\begin{cases} \mathrm{V}_\mathrm{r} = \frac{\mathrm{dr}}{\mathrm{dt}} : & \text{radial component} \\ \mathrm{V}_\theta = \mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dt}} : & \text{transverse component} \end{cases}$$

- Acceleration Vector:

Its expression is obtained by differentiating the velocity vector:

$$\begin{aligned}
\overrightarrow{\mathrm{a}} &= \frac{\mathrm{d}\overrightarrow{\mathrm{V}}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\mathrm{dr}}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\mathrm{r} + \mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\theta\right) \\
&= \frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2}\overrightarrow{\mathrm{u}}_\mathrm{r} + \frac{\mathrm{dr}}{\mathrm{dt}}\frac{\mathrm{d}\overrightarrow{\mathrm{u}}_\mathrm{r}}{\mathrm{dt}} + \frac{\mathrm{dr}}{\mathrm{dt}}\frac{\mathrm{d}\theta}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\theta + \mathrm{r}\frac{\mathrm{d}^2\theta}{\mathrm{dt}^2}\overrightarrow{\mathrm{u}}_\theta + \mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dt}}\frac{\mathrm{d}\overrightarrow{\mathrm{u}}_\theta}{\mathrm{dt}} \\
&= \frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2}\overrightarrow{\mathrm{u}}_\mathrm{r} + \frac{\mathrm{dr}}{\mathrm{dt}}\left(\frac{\mathrm{d}\theta}{\mathrm{dt}}\overrightarrow{\mathrm{u}}\theta\right) + \frac{\mathrm{dr}}{\mathrm{dt}}\frac{\mathrm{d}\theta}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\theta + \mathrm{r}\frac{\mathrm{d}^2\theta}{\mathrm{dt}^2}\overrightarrow{\mathrm{u}}_\theta + \mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dt}}\left(-\frac{\mathrm{d}\theta}{\mathrm{dt}}\overrightarrow{\mathrm{u}}_\mathrm{r}\right) \\
&= \left(\frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2} - \mathrm{r}\left(\frac{\mathrm{d}\theta}{\mathrm{dt}}\right)^2\right)\overrightarrow{\mathrm{u}}_\mathrm{r} + \left(2\frac{\mathrm{dr}}{\mathrm{dt}}\frac{\mathrm{d}\theta}{\mathrm{dt}} + \mathrm{r}\frac{\mathrm{d}^2\theta}{\mathrm{dt}^2}\right)\overrightarrow{\mathrm{u}}_\theta \\
&= \left(\ddot{\mathrm{r}} - \mathrm{r}\dot{\theta}^2\right)\overrightarrow{\mathrm{u}}_\mathrm{r} + (2\dot{\mathrm{r}}\dot{\theta} + \mathrm{r}\ddot{\theta})\overrightarrow{\mathrm{u}}_\theta \\
&= \mathrm{a}_\mathrm{r}\overrightarrow{\mathrm{u}}_\mathrm{r} + \mathrm{a}_\theta\overrightarrow{\mathrm{u}}_\theta
\end{aligned}$$

By identification, we obtain:

$$\overrightarrow{\mathrm{a}}\begin{cases} \mathrm{a}_\mathrm{r} = \ddot{\mathrm{r}} - \mathrm{r}\dot{\theta}^2 : \text{radial component} \\ \mathrm{a}_\theta = 2\dot{\mathrm{r}}\dot{\theta} + \mathrm{r}\ddot{\theta} : \text{transverse component} \end{cases}$$