

Chapter 3

WORK AND ENERGY

INTRODUCTION

In the previous chapter, dedicated to dynamics, we established the relationships between motion and the forces that cause it. In particular, we saw that knowing the forces acting on a particle and the initial conditions (position and velocity) can allow us to predict its motion. However, we do not always know all the forces at play, and even if we do, the equations to be solved can be challenging to handle. In such situations, we can turn to concepts such as work and energy, which are the focus of this chapter.

I. WORK OF A FORCE

I.1. Definitions

If a particle undergoes an infinitesimal displacement $d\vec{r}$ under the influence of a force \vec{F} , this force performs an infinitesimal work dW defined as:

$$dW = \vec{F} \cdot d\vec{r}$$

If the particle is displaced from point A to point B (figure 3.1), the total work done is

$$W(\vec{F})_A^B = \int_A^B \vec{F} \cdot d\vec{r}$$

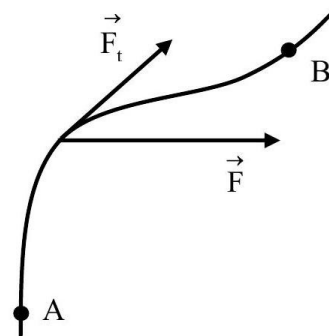


Figure 3.1

Note that

$$\vec{F} \cdot d\vec{r} = |\vec{F}| \cdot |d\vec{r}| \cos(\vec{F}, d\vec{r}) = F_t |d\vec{r}| = F_t ds$$

where F_t is the tangential component of \vec{F} and s is the curvilinear coordinate.

$$W(\vec{F})_A^B = \int_{s_A}^{s_B} F_t ds$$

Work W is expressed in Joules (1 J = 1Nm).

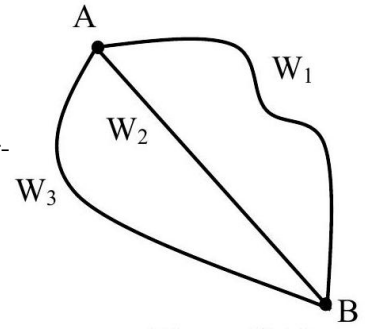


Figure 3.2

Remarks:

- If the force \vec{F} is perpendicular to the displacement, its work is zero.
- W is an algebraic quantity: if W is positive, the work is called "positive work," and if it is negative, it is called "negative work."
- Work generally depends on the path taken from A to B.

Example: In the case of figure 3.2, representing three different paths, often we have:

$$W_1(\vec{F})_A^B \neq W_2(\vec{F})_A^B \neq W_3(\vec{F})_A^B$$

1.2. Use of Cartesian Coordinates

By expressing the dot product in terms of the components of \vec{F} and $d\vec{r}$, i.e.,

$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$$

we obtain:

$$\begin{aligned} W(\vec{F})_A^B &= \int_A^B \vec{F} \cdot d\vec{r} \\ &= \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz \end{aligned}$$

Note: If the force \vec{F} is constant in magnitude and direction, the expression for $W(\vec{F})_A^B$ becomes simpler:

$$\begin{aligned} W(\vec{F})_A^B &= F_x \int_{x_A}^{x_B} dx + F_y \int_{y_A}^{y_B} dy + F_z \int_{z_A}^{z_B} dz \\ &= F_x(x_B - x_A) + F_y(y_B - y_A) + F_z(z_B - z_A) \end{aligned}$$

This result shows that work depends only on the initial and final positions and not on the path taken. The work of gravity is a good illustration.

Application:

A body with mass $m = 1$ kg is pulled with a force \vec{F} over a distance of 10 m along an inclined plane at 45° to the horizontal (figure 3.3). Assuming negligible friction and that the motion is accelerated with an acceleration of $|\vec{a}| = 1$ m/s², calculate the work done By the applied forces on the body:

The application of Newton's second law allows us to write:

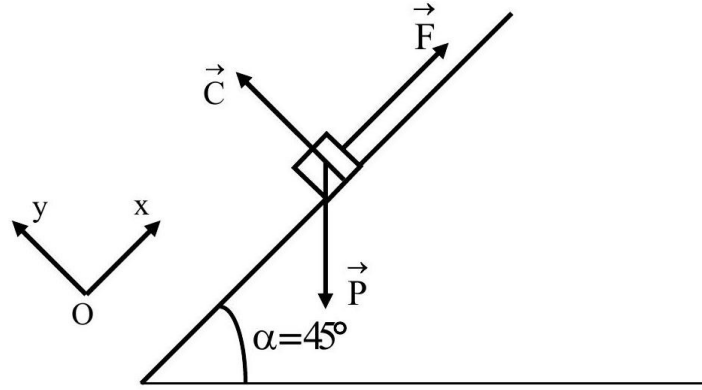


Figure 3.3

$$\vec{F} + \vec{P} + \vec{C} = m\vec{a}$$

Let's project this equation onto the two Cartesian coordinate axes:

$$\begin{aligned} \left(\sum F_{\text{ext}}\right)_x &= F - P \sin \alpha = m a \\ \left(\sum F_{\text{ext}}\right)_y &= C - P \cos \alpha = 0 \end{aligned}$$

Equation (IV.10) allows us to write:

$$F = ma + P \sin \alpha = m(a + g \sin 45^\circ) = 8.07 \text{ N}$$

The work done by \vec{F} is then:

$$\begin{aligned} W(\vec{F})_A^B &= \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy \\ &= F_x (x_B - x_A) + F_y (y_B - y_A) \\ &= F (x_B - x_A) + 0 (y_B - y_A) \\ &= 8.07 \times 10 \text{ J} \\ &= 80.7 \text{ J} \end{aligned}$$

The work done by \vec{P} is given by:

$$\begin{aligned} W(\vec{P})_A^B &= \int_{x_A}^{x_B} P_x dx + \int_{y_A}^{y_B} P_y dy \\ &= P_x (x_B - x_A) + P_y (y_B - y_A) \\ &= -P \sin \alpha (x_B - x_A) - P \cos \alpha \cdot (0) \\ &= -70.7 \text{ J} \end{aligned}$$

Since friction is negligible, the contact force \vec{C} is then perpendicular to the plane, hence to the displacement, and its work is zero.

I.3. Concept of Power

To vertically lift a body to a height h at a constant velocity, a force \vec{F} must be applied such that:

$$\vec{P} + \vec{F} = 0$$

Hence,

$$|\vec{F}| = |\vec{P}| = mg$$

and

$$W(\vec{F}) = mgh$$

Thus, since time does not appear in this expression, the value of the work is the same whether this displacement takes one second or one year. To account for the rate at which this work is done, we define power, which is the rate of change of work with respect to time:

$$p = \frac{dW}{dt}$$

Expanding the infinitesimal work, we obtain:

$$p = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{V}$$

The average power developed between two instants t_i and t_f is:

$$P_{\text{avg}} = \frac{W|_{t_i}^{t_f}}{\Delta t}$$

Power is expressed in Watts (W).

II. KINETIC ENERGY

II.1. Definition

Consider a particle moving under the action of a resultant force \vec{F} between two points A and B. Since Newton's second law is applicable along the path $A \rightarrow B$, we can use it in the expression for work. Indeed,

$$W(\vec{F})_A^B = \int_A^B \vec{F} \cdot d\vec{r}$$

Knowing that

$$\begin{cases} \vec{F} = m\vec{a} \\ d\vec{r} = \vec{V} dt \end{cases}$$

we get

$$W(\vec{F})_A^B = m \int_A^B \frac{d\vec{V}}{dt} \cdot \vec{V} dt = m \int_{V_A}^{V_B} \vec{V} \cdot d\vec{V} = \frac{1}{2} m (V_B^2 - V_A^2)$$

which means:

$$W(\vec{F})_A^B = \Delta \left(\frac{1}{2} m V^2 \right)_A^B$$

By definition, the quantity:

$$E_C = \frac{1}{2} m V^2$$

is called kinetic energy; it is measured in Joules.

II.2. Kinetic Energy Theorem

The relation

$$W(\vec{F})_A^B = \Delta \left(\frac{1}{2} m V^2 \right)_A^B = \Delta E_{C|A}^B$$

expresses the kinetic energy theorem, which can be stated as follows:

When a body moves between two points A and B under the action of a resultant force \vec{F} , the work done by this force is, regardless of the path taken and the nature of the forces, equal to the change in kinetic energy of the body.

Remarks:

- Kinetic energy is always positive.
- If the body is subjected to several forces, with resultant $\vec{F} = \sum_i \vec{F}_i$, then:

$$\Delta K_{E|A}^B = \sum_i W_i(\vec{F}_i)_A^B = W_1(\vec{F}_1)_A^B + W_2(\vec{F}_2)_A^B + W_3(\vec{F}_3)_A^B + \dots$$

- $\vec{p} = m\vec{V}$ being the momentum, we have:

$$P^2 = m^2 V^2 \Rightarrow E_c = \frac{P^2}{2m}$$

- If \vec{F} is perpendicular to the displacement, its work is zero, and consequently, the kinetic energy is constant.

III. CONSERVATIVE FORCES AND POTENTIAL ENERGY

III.1. Conservative Forces

A force \vec{F} is called conservative if it satisfies one of these two equivalent conditions:

First condition: the work done by the force \vec{F} between two points A and B does not depend on the path taken. This independence of the path followed implies, for Figure 3.4.

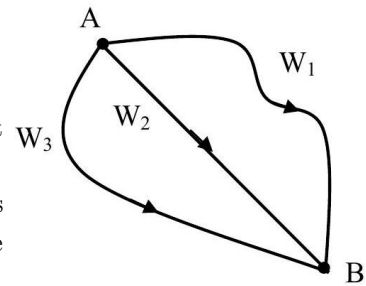


Figure 3.4

$$W_1(\vec{F})_A^B = W_2(\vec{F})_A^B = W_3(\vec{F})_A^B$$

So,

$$\int_A^B \vec{F} \cdot d\vec{r}_1 = \int_A^B \vec{F} \cdot d\vec{r}_2 = \int_A^B \vec{F} \cdot d\vec{r}_3$$

Second condition: the total work over a closed path (i.e., a round trip) is zero.

Indeed, due to the independence of work with respect to the path taken:

$$\int_A^B \vec{F} \cdot d\vec{r}_1 = \int_A^B \vec{F} \cdot d\vec{r}_2 \quad \Rightarrow \quad \int_A^B \vec{F} \cdot d\vec{r}_1 - \int_A^B \vec{F} \cdot d\vec{r}_2 = 0$$

which can be written as

$$\int_A^B \vec{F} \cdot d\vec{r}_1 + \int_B^A \vec{F} \cdot d\vec{r}_2 = 0 \quad \text{or} \quad W_1(\vec{F})_A^B + W_2(\vec{F})_B^A = 0$$

Remarks:

- The work done by a conservative force, as it does not depend on the path, must depend only on position and not on velocity or time.
- Examples of conservative forces include elastic, gravitational, and electric forces.

III.2. Concept of Potential Energy

a. First Experiment

Consider the mass-spring system in Figure 3.5, in the absence of friction with the ground. We displace the mass to compress the spring by a length a , and then we release the system. The mass then undergoes oscillations, with an amplitude of a , centered at point O. The left part of the figure represents characteristic configurations of this motion over one cycle.

At the extreme points, i.e., at $x = \pm a$, the mass has no kinetic energy since its velocity is zero. However, it starts moving back towards the equilibrium position $x = 0$ with increasing velocity. This means it gains kinetic energy. Since energy is conserved, it must come from the transformation of a form of energy due to the state (compression or extension) of the mass-spring system. This energy is called potential energy (symbol P_E).

Beyond the equilibrium point, the mass undergoes deceleration until the next extreme point, during which it loses all its kinetic energy. In fact, the system regains potential energy that it uses to send the mass back to the equilibrium position. This motion is maintained by a succession of transformations from potential energy to kinetic energy and from kinetic energy to potential energy. This is true when friction is absent.

Note that the force responsible for this motion is the elastic force, which is conservative (weight and contact force cancel out). Therefore, the associated potential energy is referred to as elastic potential energy. Furthermore, the experiment is reproducible, meaning that when repeated, the same results are obtained. Moreover, if the amplitude "a" is changed, the potential energy will vary quantitatively. This means that potential energy depends on the position occupied by the mass.

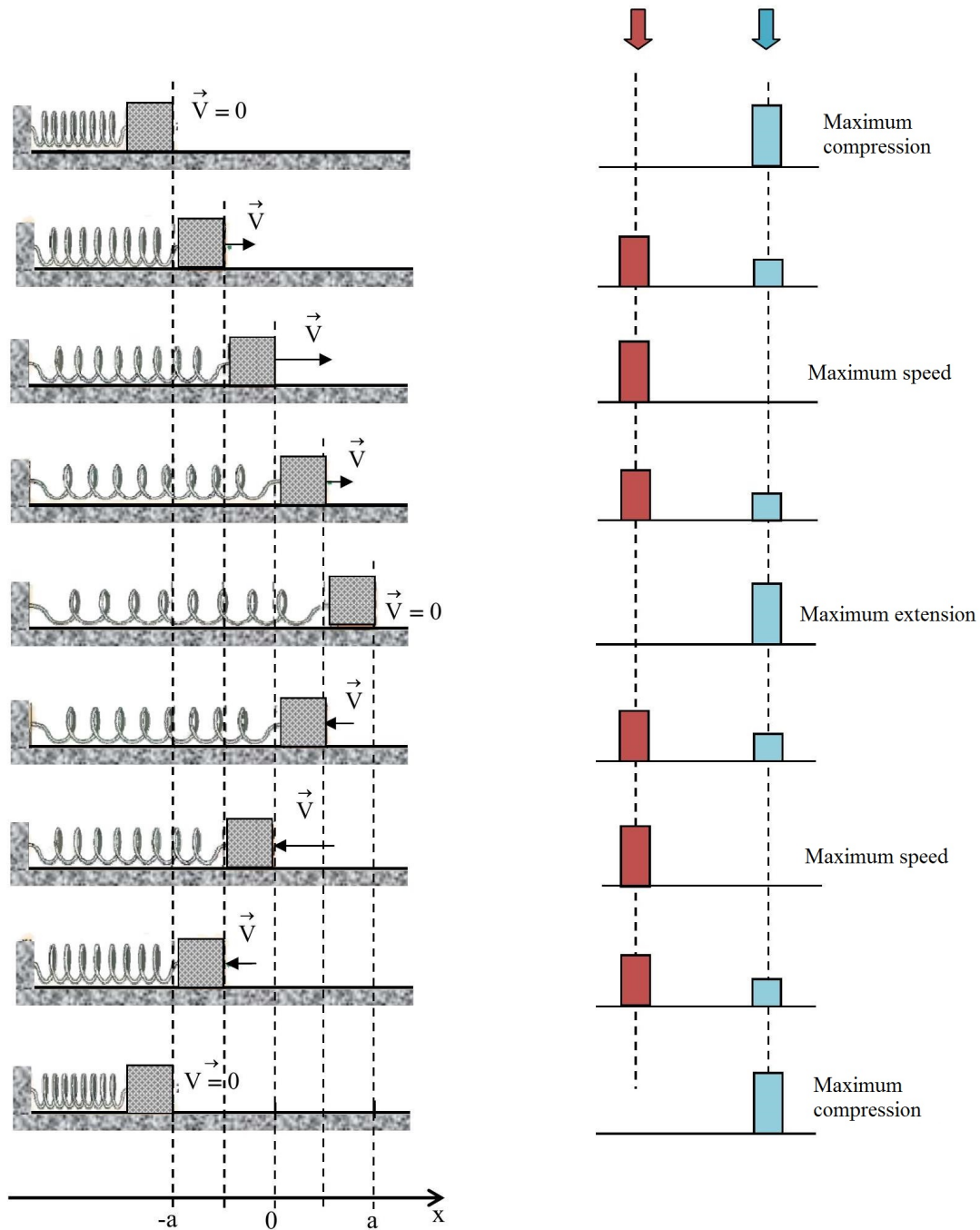


Figure 3.5

b. Second Experiment

Consider the free fall of an object dropped from a height h above the ground (Figure 3.6). The object begins to move under the influence of its weight, a conservative force exerted by the Earth. Initially, the object has no kinetic energy since its velocity is zero. Therefore, the system's state (Earth-object) means that it possesses gravitational potential energy, which can be transformed into kinetic energy. Since the experiment is reproducible, it means that the body recovers its potential energy each time it returns to its initial position. By changing the height of the fall and studying the resulting kinetic energy, it becomes clear that this potential energy depends on the object's position.

Now, consider the case where the same body is launched upward from the ground with an initial velocity \vec{V}_0 , resulting in a kinetic energy of $mV_0^2/2$. When it reaches a certain height h , its velocity

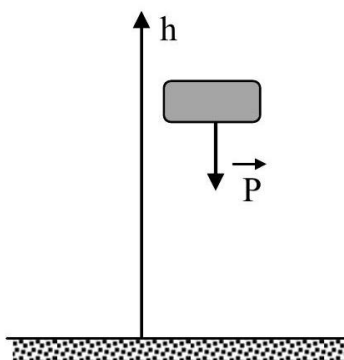


Figure 3.6

becomes zero (having lost all of its kinetic energy), and it then undergoes free fall similar to the one described earlier. Thus, between the ground and the turning point, the body transforms its kinetic energy into potential energy, which it then reconverts into kinetic energy as it falls back.

III.3. Total Mechanical Energy

We have seen that the potential energy E_P of a body is energy due to the state of the system responsible for the conservative force it experiences. This energy, which depends on the position occupied by the body, can be transformed into kinetic energy E_C , and the body recovers it when it returns to its initial state. Thus, the system can transform potential energy into kinetic energy and vice versa. Consider the displacement of a particle under the influence of a conservative force:

- if the body has lost an amount of potential energy $\Delta E_P (< 0)$, it gains an amount of kinetic energy $\Delta E_C (> 0)$;
- if the body has lost an amount of kinetic energy $\Delta E_C (< 0)$, it gains an amount of potential energy $\Delta E_P (> 0)$.

Considering the signs of both quantities, in all cases, we have the equality:

$$\Delta E_P = -\Delta E_C$$

This allows us to write:

$$\Delta E_P + \Delta E_C = 0 \Leftrightarrow \Delta (E_P + E_C) = 0$$

or

$$E_P + E_C = \text{constant}$$

The quantity $E_T = E_P + E_C$ represents the total mechanical energy of the particle.

Theorem: the total mechanical energy of a particle subjected to conservative forces is constant.

III.4. Determination of Potential Energy

a. Method

Consider a body moving between two positions, $A_0(\vec{r}_0)$ and $A(\vec{r})$, under the influence of a conservative force \vec{F} . The theorem of mechanical energy allows us to write:

$$\Delta E_P|_{A_0}^A = - \Delta E_C|_{A_0}^A$$

Due to the theorem of kinetic energy,

$$\Delta E_P|_{A_0}^A = - \Delta K_C|_{A_0}^A = -W(\vec{F})_{A_0}^A = - \int_{A_0}^A \vec{F} \cdot d\vec{r}$$

So,

$$P_E(A) - P_E(A_0) = - \int_{A_0}^A \vec{F} \cdot d\vec{r}$$

Thus, to determine the potential energy at any point A, one must know that of a reference point A_0 . Often, A_0 is chosen, by convention, where the force \vec{F} is zero (whenever possible), and we arbitrarily assign zero value to $E_P(A_0)$.

b. Elastic Potential Energy

Let's return to the mass-spring system from paragraph III.2.a. Representing the deformation of the spring by x , the elastic force applied by the spring is:

$$\vec{F} = -kx\vec{i}$$

This force vanishes at the origin ($x = 0$), which we will use as the reference point. Consequently,

$$P_E(x) - \underbrace{P_E(0)}_0 = - \int_0^x F \cdot dx = - \int_0^x (-kx) \cdot dx = \frac{1}{2}kx^2$$

In general, the potential energy associated with an elastic force $\vec{F} = -k\vec{\Delta l}$, in the case of a deformation $\vec{\Delta l}$, is given by

$$P_E(\Delta l) = \frac{1}{2}k(\Delta l)^2$$

c. Gravitational Potential Energy

Consider a body of mass m moving under the influence of the gravitational force \vec{F}_g exerted by the Earth on it (figure 3.7), which is:

$$\vec{F}_g = -G \frac{M_T m}{r^2} \vec{u}$$

It's evident that this force vanishes at infinity, which we will consider as the reference:

$$P_E(r) = - \int_{\infty}^r \vec{F}_g \cdot d\vec{r} = - \int_{\infty}^r \left(-G \frac{M_T m}{r^2} \right) \vec{u} \cdot d\vec{r}$$

Note that

$$\vec{u} \cdot d\vec{r} = |\vec{u}| |d\vec{r}| \cos(\vec{u}, d\vec{r}) = dr$$

Hence,

$$E_P(r) = -GM_T m \int_{\infty}^r \left(-\frac{1}{r^2} \right) dr = -G \frac{M_T m}{r}$$

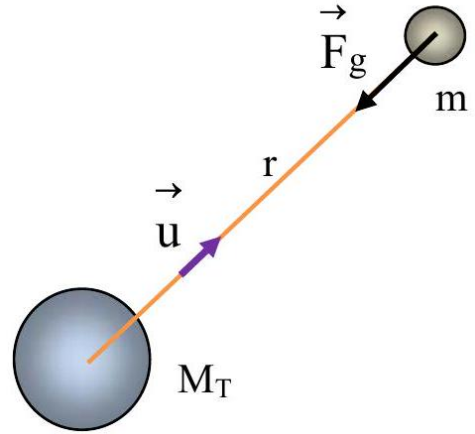


Figure 3.7

d. Gravitational Potential Energy in the Vicinity of Earth (or Gravitational Potential Energy)

Consider a body of mass m moving under the influence of its weight between two points A and B. The variation of its potential energy is obtained by writing:

$$\Delta E_P|_A^B = - \Delta E_C|_A^B = -W(\vec{P})_A^B$$

The work done by \vec{P} is given by

$$\begin{aligned} W(\vec{P})_A^B &= \int_{x_A}^{x_B} P_x dx + \int_{y_A}^{y_B} P_y dy \\ &= P_x (x_B - x_A) + P_y (y_B - y_A) \end{aligned}$$

In a Cartesian coordinate system with the (Oy) axis ascending vertically, the components of \vec{P} are: $P_x = 0$; $P_y = -mg$. Consequently,

$$\Delta E_P|_A^B = -W(\vec{P})_A^B = mg (y_B - y_A)$$

This result holds for any points A and B; therefore, we can generally write:

$$\Delta E_P = mg \Delta y$$

By convention, we assign the value zero to E_P at $y = 0$. The potential energy is then expressed as:

$$P_E(y) = mgy$$

It is possible to choose the reference level for potential energy ($E_P = 0$) at any altitude. Thus, introducing an ascending vertical axis with its origin at this reference level, the gravitational potential energy is written as:

$$P_E(h) = mgh$$

where h is the coordinate of the position on this axis.

Example: Consider a body moving on a trajectory in the shape of a quarter-circle of radius R situated in the vertical plane (O, x, y) . We aim to express the gravitational potential energies of the object for the two positions M and N and the two reference levels O and O' in figure 3.8.

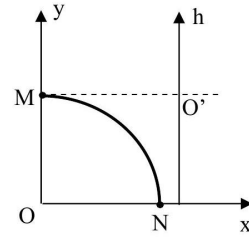


Figure 3.8

Position	Reference: O	Reference: O'
M	mgR	0
N	0	$-mgR$

e. Study of Some Examples

For a particle moving under the influence of conservative forces, the total energy remains constant. To calculate it, we need to find a position where the velocity is known.

a. Particle in an Elastic Force Field

Let's return to the mass-spring system from paragraph III.2.a. Representing the spring's deformation as x , the potential energy of the mass is given by

$$E_P(x) = \frac{1}{2}kx^2$$

Its total energy is then

$$E_T = \frac{1}{2}kx^2 + \frac{1}{2}mV^2$$

Knowing that the velocity is zero at the maximum elongation point, $x = a$, we obtain

$$E_T = \frac{1}{2}ka^2$$

Hence, at any point with abscissa x ,

$$E_T = \frac{1}{2}kx^2 + \frac{1}{2}mV^2 = \frac{1}{2}ka^2$$

Thus, the kinetic energy at this point is:

$$E_C(x) = \frac{1}{2}mV^2 = E_T - E_P = \frac{1}{2}k(a^2 - x^2)$$

Consequently, the velocity at a point with abscissa x is given by the relation

$$V = \sqrt{\frac{k}{m}(a^2 - x^2)}$$

The energy diagram in figure 3.9 below clearly shows that any decrease in potential energy is accompanied by an increase in an equivalent amount of kinetic energy.

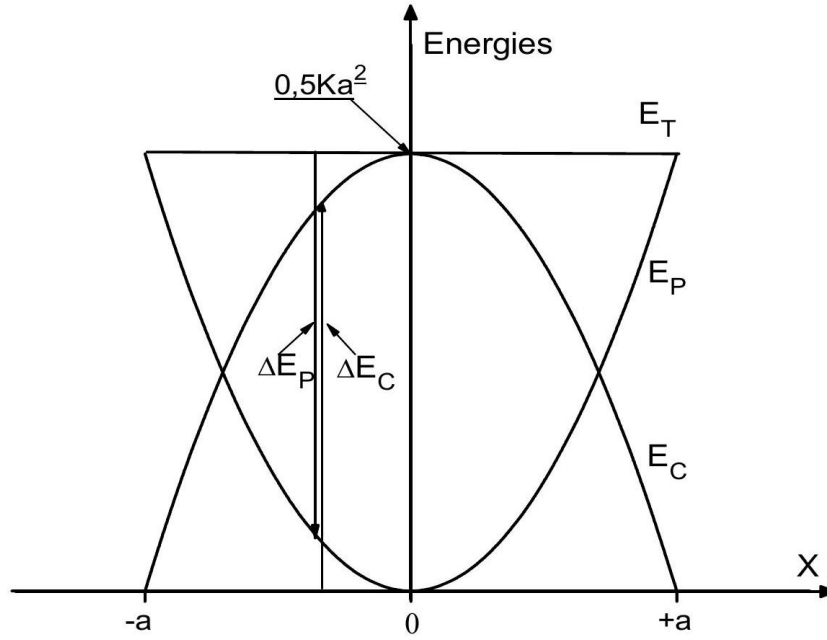


Figure 3.9

b. Projectile Motion Launched Skyward from the Ground: Escape Velocity

Let \vec{V}_0 be the initial velocity of a projectile launched skyward. Its gravitational potential energy is given by the expression:

$$P_E(r) = -G \frac{M_T m}{r}$$

where r is the distance from the center of the Earth, m is its mass, and M_T is the mass of the Earth. Its total mechanical energy is then

$$E_T = -G \frac{M_T m}{r} + \frac{1}{2} m V^2 = \text{constant}$$

Using the initial conditions $r = R_T$ (the radius of the Earth) and $V = V_0$, we get

$$E_T = -G \frac{M_T m}{r} + \frac{1}{2} m V^2 = -G \frac{M_T m}{R_T} + \frac{1}{2} m V_0^2$$

The kinetic energy at any point of the trajectory is then

$$E_C = \frac{1}{2} m V^2 = G \frac{M_T m}{r} - G \frac{M_T m}{R_T} + \frac{1}{2} m V_0^2$$

The diagram of the different energies is shown in figure 3.10.

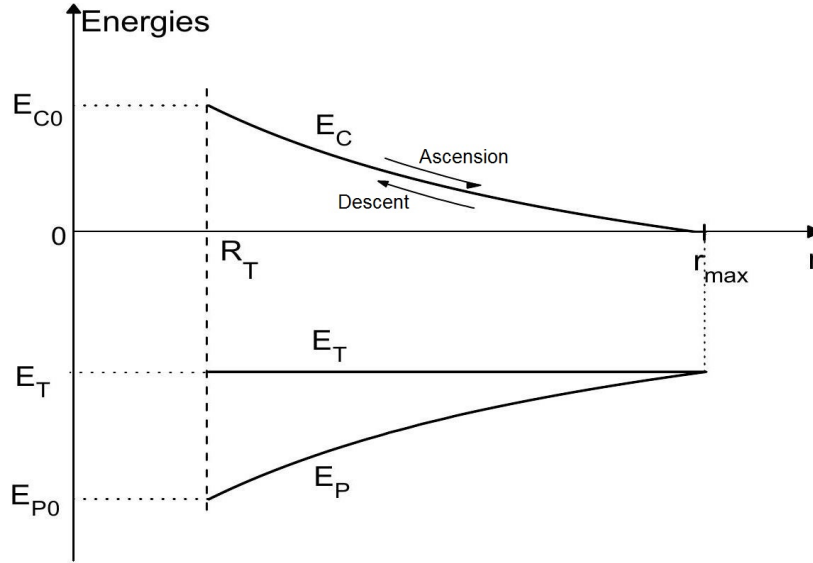


Figure 3.10

The distance r_{\max} corresponds to the point of maximum altitude reached before entering free fall returning the body to the Earth. For the projectile to escape the Earth's attraction, assuming it's solely under the effect of Earth's gravity, its initial velocity would allow it to go to infinity where its kinetic energy would be zero. Its total mechanical energy is then

$$E_T = \underbrace{-G \frac{M_T m}{r_\infty}}_0 + \underbrace{\frac{1}{2} m V_\infty^2}_0 = -G \frac{M_T m}{R_T} + \frac{1}{2} m V_0^2 = 0$$

Hence, the escape velocity of the projectile is

$$V_0 = \sqrt{2G \frac{M_T}{R_T}}$$

III.5. Conservative Forces and Potential Energy

Let's consider the movement of a particle constrained to move under the influence of a conservative force \vec{F} . In this case, its change in potential energy between two points A_0 and A_1 is given by the relation

$$\Delta E_P|_{A_0}^{A_1} = - \int_{A_0}^{A_1} \vec{F} \cdot d\vec{r}$$

For an elemental displacement, this relation becomes

$$dE_P = -\vec{F} \cdot d\vec{r} = -F_x dx - F_y dy - F_z dz$$

Moreover, if a function $f(x, y, z)$ is continuously differentiable, meaning the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ exist and are continuous, its gradient is defined, denoted as $\overrightarrow{\text{grad}} f$, as the vector whose components are the partial derivatives of this function, i.e.,

$$\overrightarrow{\text{grad}} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

The difference

$$df = f(x + dx, y + dy, z + dz) - f(x, y, z)$$

a. Determination of Force from the $E_P(x)$ Diagram

The conservative force and the associated potential energy are related by:

$$\vec{F} = -\overrightarrow{\text{grad}}E_P = -\vec{\nabla}E_P$$

in our case

$$F_x(x) = -\frac{d_p(x)}{dx} = - \text{the slope of the tangent to the graph of } P_E(x)$$

Thus, we can easily determine $F_x(x)$ by graphically evaluating the slope of the tangent to the potential energy diagram.

b. Determination of Equilibrium Positions

In our case, the force field is a simple function $F(x)$, and the force points to the right if $F_x > 0$, to the left if $F_x < 0$. A point with abscissa x_0 is an equilibrium point if the force is zero at that location $F(x_0) = 0$; in other words, if the derivative of the potential energy vanishes at x_0 . It's a stable equilibrium where if the particle is placed at this point without initial velocity, it would remain stationary.

An equilibrium is termed stable if, when the particle is slightly displaced from x_0 , the force tends to bring it back to this position. This corresponds to a minimum in the potential energy diagram. This condition is fulfilled if the first derivative of the potential energy is zero and its second derivative is positive at point x_0 (concave upward). The studied potential energy diagram reveals two stable equilibrium points: $C(x = -2m)$ and $G(x = 3 m)$.

In the case where x_0 corresponds to a maximum in the potential energy diagram, the force tends to move the particle away from x_0 if the latter is slightly displaced, and the equilibrium is termed unstable. The first derivative of the potential energy is zero, and its second derivative is negative (concave downward). In the diagram, there's only one unstable equilibrium point: the origin O .

c. Determination of Turning Points

Turning points (or limit points) are the points where kinetic energy becomes zero. To find them, we use the relation

$$E_C = \frac{1}{2}mVV^2 = E_T - E_P$$

These points where the total and potential energies are equal define the particle's trajectory. Indeed, kinetic energy being a positive quantity, relation (IV.69) indicates that motion can only occur in the region $E_P(x) \leq E_T$. For example, if $E_T = -10$ J, the potential energy diagram shows that points F and H are turning points. The particle cannot move to the left of F or to the right of H where the potential energy is higher than $E_T = -10$ J. This scenario might occur, for instance, if initially released at F or H without initial velocity.

$$E_T = E_{P0} + E_{C0} = -10 + 0 = -10 \text{ J}$$

It will perform an oscillatory movement between these two points according to the scheme (figure 3.12).

The portion (F'G'H') of the curve is called a **potential well** because the particle cannot leave the segment (FH).

If $E_T = -2.5$ J, we will have four turning points (B, D, E, I) and two potential wells: (B'C'D') and (E'G'I'). The particle will oscillate in either of the two wells but cannot pass from one to the other due to the portion (D'OE') of the potential energy curve called the **potential barrier**. For example:

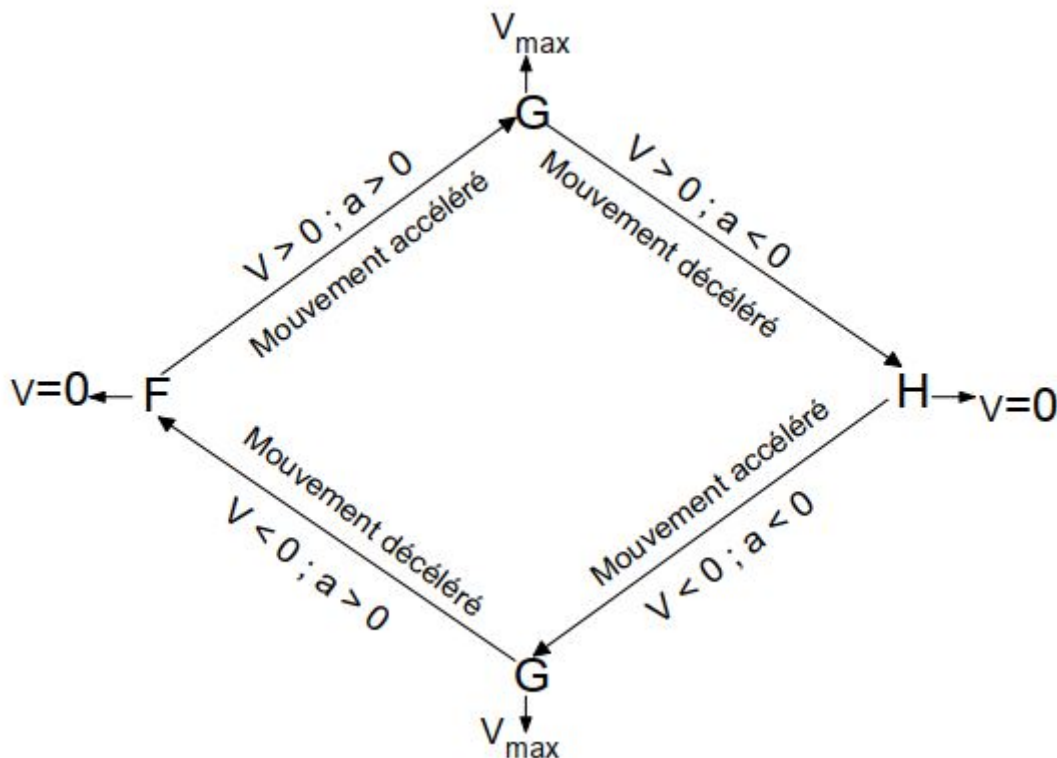


Figure 3.12. Caption

- If initially released at B or D without initial velocity, it will oscillate between B and D.
- If initially released at E or I without initial velocity, it will oscillate between E and I.
- If initially launched from F to the right with a kinetic energy $E = 7.5$ J, its total energy will be:
 $E_C = E_T + E_{P0} + E_{C0} = -2.5 + 10 = 7.5$ J.

The diagram reveals that to satisfy the condition $P_E(x) \leq E_T$, it will reach point I and then oscillate between E and I. In the case $E_T = 5$ J, we will have two turning points, A and J, and one potential well.

The portion (F’G’H’) of the curve is called a potential well because the particle cannot leave the segment (FH). If $E_T = -2.5$ J, we will have four turning points (B, D, E, I) and two potential wells: (B’C’D’) and (E’G’I’). The particle will oscillate in either of the two wells but cannot pass from one to the other due to the portion (D’OE’) of the potential energy curve called the potential barrier. For example:

- If initially released at B or D without initial velocity, it will oscillate between B and D.
- If initially released at E or I without initial velocity, it will oscillate between E and I.
- If initially launched from F to the right with a kinetic energy $E = 7.5$ J, its total energy will be:
 $E_C = E_T + E_{P0} + E_{C0} = -2.5 + 10 = 7.5$ J.

The diagram reveals that to satisfy the condition $P_E(x) \leq E_T$, it will reach point I and then oscillate between E and I. In the case $E_T = 5$ J, we will have two turning points, A and J, and one potential well.

In the case where the particle is released without initial velocity at point A_0 , its total energy will have a value of 10 J, and it will move in the positive x direction as long as its potential energy remains below this value ($E_p < 10$ J).

Recall that for a particle moving under the influence of conservative forces,

$$\begin{cases} \Delta E_T = \Delta P_E + \Delta E_C = 0 \\ \Delta E_C = \sum W(\vec{F}_C) \end{cases} \Rightarrow \Delta P_E + \sum W(\vec{F}_C) = 0$$

where $\sum W(\vec{F}_C)$ is the work done by conservative forces.

If the particle is subjected to both conservative and non-conservative forces,

$$\begin{cases} \Delta E_T = \Delta P_E + \Delta E_C \\ \Delta E_C = \sum W(\vec{F}_C) + \sum W(\vec{F}_{NC}) \end{cases} \Rightarrow \Delta E_T = \underbrace{\Delta P_E + \sum W(\vec{F}_C)}_0 + \sum W(\vec{F}_{NC})$$

thus

$$\Delta E_T = \sum W(\vec{F}_{NC})$$

$\sum W(\vec{F}_{NC})$ represents the work done by non-conservative forces.

Theorem: The change in the total mechanical energy of a body is equal to the work done by the resultant non-conservative forces acting on it.

As examples, we mention the following non-conservative forces:

- Contact force (except for the elastic force);
- Magnetic force;
- All forces explicitly dependent on time or velocity.

Note: The non-conservative force that often appears in exercises is the contact force denoted \vec{C} . Between two points A and B on the trajectory followed by the body on which it applies, its work is:

$$W(\vec{C})_A^B = W(\vec{C}_{II})_A^B + \underbrace{W(\vec{C}_{\perp})_A^B}_0$$

hence

$$W(\vec{C})_A^B = \int_A^B \vec{C}_{\parallel} \cdot d\vec{r} = - \int_{s_A}^{s_B} |\vec{C}_{II}| \cdot ds$$

In the specific case where $|\vec{C}_{II}|$ remains constant during the motion, we will have:

$$W(\vec{C})_A^B = - |\vec{C}_{\parallel}| \int_{s_A}^{s_B} ds = - |\vec{C}_I| (s_B - s_A)$$